Strong PNT

Math Inc.

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Chapter 1

Complex Analysis

Lemma 1 (Log growth). *For $t > 1$ we have $2\log t \leq O(\log t)$*

Proof. Uses definition of big $O(.)$ □

Lemma 2 (Square log). *For $t \geq 2$ we have $\log(t^2) = 2\log t$*

Proof. □

Lemma 3 (Double log). *For $t \geq 2$ we have $\log(2t) \leq \log(t^2)$*

Proof. Uses $2t \leq t^2$, and $\log(t)$ monotonically increasing. □

Lemma 4 (Log compare). *For $t \geq 2$ we have $\log(2t) \leq 2\log t$*

Proof. Apply Lemmas 2 and 3. □

Lemma 5 (Exp rule). *For any $n \geq 1$ and $\alpha, \beta \in \mathbb{C}$ we have $n^{\alpha+\beta} = n^\alpha \cdot n^\beta$*

Proof. □

Lemma 6 (Real scale). *For $b \in \mathbb{R}$ and $w \in \mathbb{C}$ we have $\Re(bw) = b\Re(w)$.*

Proof. □

Lemma 7 (Real series). *For a convergent series $v = \sum_{n=1}^{\infty} v_n$ with $v_n \in \mathbb{C}$, we have $\Re(v) = \sum_{n=1}^{\infty} \Re(v_n)$.*

Proof. □

Lemma 8 (Euler's formula). *For $a \in \mathbb{R}$ we have $e^{ia} = \cos(a) + i\sin(a)$*

Proof. □

Lemma 9 (Real cosine). *For $a \in \mathbb{R}$ we have $\Re(e^{ia}) = \cos(a)$*

Proof. Apply Lemma 8. □

Lemma 10 (Log inverse). *For $n \geq 1$ we have $n = e^{\log n}$.*

Proof. □

Lemma 11 (Cos even)**.** *For* $a \in \mathbb{R}$ *we have* $\cos(-a) = \cos(a)$

Proof. □

Lemma 12 (Cos even)**.** *For* $n \geq 1$, $y \in \mathbb{R}$ *we have* $\cos(-y \log n) = \cos(y \log n)$

Proof. Let $a = y \log n$. Since $n \geq 1$ we have $\log n \geq 0$, so $a \in \mathbb{R}$. Apply Lemma 11 with $a = y \log n$. □

Lemma 13 (Exp form)**.** *For* $n \geq 1$ *and* $y \in \mathbb{R}$ *we have* $n^{-iy} = e^{-iy \log n}$.

Proof. By Lemma 10 $n^{-iy} = (e^{\log n})^{-iy}$. Then $(e^{\log n})^{-iy} = e^{-iy \log n}$ so $n^{-iy} = e^{-iy \log n}$. □

Lemma 14 (Real cosine)**.** *For* $n \geq 1$ *and* $y \in \mathbb{R}$ *we have* $\Re(e^{-iy \log n}) = \cos(-y \log n)$.

Proof. Let $a = -y \log n$ so $e^a = e^{-iy \log n}$. Apply Lemma 9 with $a = -y \log n$. □

Lemma 15 (Real cosine)**.** *For* $n \geq 1$ *and* $y \in \mathbb{R}$ *we have* $\Re(n^{-iy}) = \cos(-y \log n)$.

Proof. Apply Lemmas 13 and 14. □

Lemma 16 (Real cosine)**.** *For* $n \geq 1$ *and* $y \in \mathbb{R}$ *we have* $\Re(n^{-iy}) = \cos(y \log n)$.

Proof. Apply Lemmas 12 and 15. □

Lemma 17 (Double angle)**.** *For any* $\theta \in \mathbb{R}$ *we have* $\cos(2\theta) = 2\cos(\theta)^2 - 1$.

Proof. □

Lemma 18 (Cos square)**.** *For any* $\theta \in \mathbb{R}$ *we have* $2\cos(\theta)^2 = 1 + \cos(2\theta)$.

Proof. Apply Lemma 17 □

Lemma 19 (Square expand)**.** *For any* $\theta \in \mathbb{R}$ *we have* $2(1 + \cos(\theta))^2 = 2 + 4\cos(\theta) + 2\cos(\theta)^2$.

Proof. We calculate $2(1 + \cos(\theta))^2 = 2(1 + 2\cos(\theta) + \cos(\theta)^2) = 2 + 4\cos(\theta) + 2\cos(\theta)^2$. □

Lemma 20 (Trig identity)**.** *For any* $\theta \in \mathbb{R}$ *we have* $2(1 + \cos(\theta))^2 = 3 + 4\cos(\theta) + \cos(2\theta)$.

Proof. Apply Lemmas 18 and 19. □

Lemma 21 (Square nonneg)**.** *For* $y \in \mathbb{R}$ *we have* $0 \leq y^2$.

Proof. □

Lemma 22 (Double square)**.** *For* $y \in \mathbb{R}$ *we have* $0 \leq 2y^2$.

Proof. Apply Lemma 21. □

Lemma 23 (Cos square)**.** *For any* $\theta \in \mathbb{R}$ *we have* $0 \leq 2(1 + \cos(\theta))^2$.

Proof. Apply Lemma 22 with $y = 1 + \cos(\theta)$. □

Lemma 24 (Trig positive)**.** *For any* $\theta \in \mathbb{R}$ *we have* $0 \leq 3 + 4\cos(\theta) + \cos(2\theta)$.

Proof. Apply Lemmas 20 and 23. □

Lemma 25 (Trig positive)**.** *For* $n \geq 1$ *and* $t \in \mathbb{R}$ *we have* $0 \leq 3 + 4\cos(t \log n) + \cos(2t \log n)$.

Proof. Apply Lemma 24 with $\theta = t \log n$. □

Lemma 26 (Series positive)**.** *For a convergent series* $r = \sum_{n=1}^{\infty} r_n$*, if* $r_n \geq 0$ *for all* $n \geq 1$*, then* $r \geq 0$*.*

Proof. □

Lemma 27 (Real part diff)**.** *For any* $w \in \mathbb{C}$*,* $\Re(2M - w) = 2M - \Re(w)$*.*

Proof. □

Lemma 28 (Real part 2M)**.** *We have* $\Re(2M - f(z)) = 2M - \Re(f(z))$*.*

Proof. Apply Lemma 27 with $w = f(z)$. □

Lemma 29 (Inequality reversal)**.** *For* $x, M \in \mathbb{R}$*, if* $x \leq M$ *then* $2M - x \geq M$*.*

Proof. □

Lemma 30 (Real part lower bound)**.** *For* $w \in \mathbb{C}$ *and* $M > 0$*, if* $\Re(w) \leq M$ *then* $2M - \Re(w) \geq M$*.*

Proof. Apply Lemma 29 with $x = \Re(w)$. □

Lemma 31 (Real part bound)**.** *For* $w \in \mathbb{C}$ *and* $M > 0$*, if* $\Re(w) \leq M$ *then* $\Re(2M - w) \geq M$*.*

Proof. Apply Lemmas 28 and 30. □

Lemma 32 (Real part >0)**.** *For* $w \in \mathbb{C}$ *and* $M > 0$*, if* $\Re(w) \leq M$ *then* $\Re(2M - w) > 0$*.*

Proof. Apply Lemmas 28 and 30 with $w = f(z)$. □

Lemma 33 (Pos real nonzero)**.** *If* $w \in \mathbb{C}$ *has* $\Re(w) > 0$*, then* $w \neq 0$*.*

Proof. □

Lemma 34 (2M minus nonzero)**.** *For* $w \in \mathbb{C}$ *and* $M > 0$*, if* $\Re(w) \leq M$ *then* $2M - w \neq 0$*.*

Proof. Apply Lemmas 32 and 33. □

Lemma 35 (Absolute value positive)**.** *Let* $z \in \mathbb{C}$*. If* $z \neq 0$ *then* $|z| > 0$*.*

Proof. □

Lemma 36 (2M minus mod)**.** *For* $w \in \mathbb{C}$ *and* $M > 0$*, if* $\Re(w) \leq M$ *then* $|2M - w| > 0$*.*

Proof. Apply Lemmas 34 and 35 with $z = 2M - w$. □

Lemma 37 (Real imaginary)**.** *For any* $w \in \mathbb{C}$*, we have* $w = \Re(w) + i\Im w$*.*

Proof. □

Lemma 38 (Mod square)**.** *For any* $a, b \in \mathbb{R}$*, we have* $|a + ib|^2 = a^2 + b^2$*.*

Proof. □

Lemma 39 (Shifted mod)**.** *For any* $a, b, c \in \mathbb{R}$*, we have* $|c - a - ib|^2 = (c - a)^2 + b^2$*.*

Proof. □

Lemma 40 (Mod diff)**.** *For any* $a, b, c \in \mathbb{R}$*, we have* $|c - a - ib|^2 - |a + ib|^2 = (c - a)^2 - a^2$*.*

Proof. Apply Lemmas 39 and 38. □

Lemma 41 (Square expand)**.** *For any* $a, c \in \mathbb{R}$*, we have* $(c-a)^2 = a^2 - 2ac + c^2$*.*

Proof. □

Lemma 42 (Square diff)**.** *For any* $a, c \in \mathbb{R}$*, we have* $(c-a)^2 - a^2 = 2c(c-a)$*.*

Proof. Apply Lemma 41 □

Lemma 43 (Mod diff)**.** *For any* $a, b, c \in \mathbb{R}$*, we have* $|c - a - ib|^2 - |a + ib|^2 = 2c(c-a)$*.*

Proof. Apply Lemmas 40 and 42. □

Lemma 44 (Modulus diff)**.** *For any* $w \in \mathbb{C}$*,* $|2M - \Re(w) - i\Im w|^2 - |\Re(w) + i\Im w|^2 = 4M(M - \Re(w))$*.*

Proof. Apply Lemma 43 with $a = \Re(w)$ and $b = \Im w$ and $c = 2M$. □

Lemma 45 (Modulus identity)**.** *For any* $w \in \mathbb{C}$*,* $|2M - w|^2 - |w|^2 = 4M(M - \Re(w))$*.*

Proof. Apply Lemmas 44 and 37 □

Lemma 46 (Nonneg product)**.** *If* $M > 0$ *and* $x \leq M$*, then* $4M(M - x) \geq 0$*.*

Proof. □

Lemma 47 (Nonneg product)**.** *Let* $M > 0$ *and* $w \in \mathbb{C}$*. If* $\Re(w) \leq M$ *then* $4M(M - \Re(w)) \geq 0$*.*

Proof. Apply Lemma 46 with $x = \Re(w)$. □

Lemma 48 (Modulus compare)**.** *Let* $M > 0$ *and* $w \in \mathbb{C}$*. If* $\Re(w) \leq M$ *then* $|2M - w|^2 - |w|^2 \geq 0$*.*

Proof. Apply Lemmas 45 and 47 □

Lemma 49 (Modulus bound)**.** *Let* $M > 0$ *and* $w \in \mathbb{C}$*. If* $\Re(w) \leq M$ *then* $|2M - w|^2 \geq |w|^2$*.*

Proof. Apply Lemma 48. □

Lemma 50 (Modulus bound)**.** *Let* $M > 0$ *and* $w \in \mathbb{C}$*. If* $\Re(w) \leq M$ *then* $|2M - w| \geq |w|$*.*

Proof. Apply Lemma 49 and take non-negative square-root. □

Lemma 51 (Modulus order)**.** *Let* $M > 0$ *and* $w \in \mathbb{C}$*. If* $\Re(w) \leq M$ *then* $|w| \leq |2M - w|$*.*

Proof. Apply Lemma 50. □

Lemma 52 (Divide inequality)**.** *If* $c > 0$ *and* $0 \leq a \leq b$*, then* $a/c \leq b/c$*.*

Proof. □

Lemma 53 (Ratio bound)**.** *If* $b > 0$ *and* $0 \leq a \leq b$*, then* $a/b \leq 1$*.*

Proof. Apply Lemma 52 with $c = b > 0$. □

Lemma 54 (Ratio bound)**.** *Let* $M > 0$ *and* $w \in \mathbb{C}$*. If* $|2M - w| > 0$ *and* $|w| \leq |2M - w|$ *then* $\frac{|w|}{|2M-w|} \leq 1$*.*

Proof. Apply Lemmas 36 and 51 and 53 with $a = |w|$ and $b = |2M - w|$. □

Lemma 55 (Ratio bound)**.** *Let $M > 0$ and $w \in \mathbb{C}$. If $\Re(w) \le M$ and $|w| \le |2M - w|$ then $\frac{|w|}{|2M-w|} \le 1$.*

Proof. Apply Lemmas 36 and 54. □

Lemma 56 (Ratio bound)**.** *Let $M > 0$ and $w \in \mathbb{C}$. If $\Re(w) \le M$ then $\frac{|w|}{|2M-w|} \le 1$.*

Proof. Apply Lemmas 51 and 55. □

Lemma 57 (Triangle inequality)**.** *Let $N, G \in \mathbb{C}$. We have $|N + G| \le |N| + |G|$*

Proof. □

Lemma 58 (Triangle minus)**.** *Let $N, F \in \mathbb{C}$. We have $|N - F| \le |N| + |F|$*

Proof. Apply Lemma 57 with $G = -F$. □

Lemma 59 (Scaled triangle)**.** *Let $r > 0$ and $N, F \in \mathbb{C}$. We have $r|N - F| \le r(|N| + |F|)$*

Proof. Apply Lemmas 57 and 52 with $a = |N - F|$ and $b = (|N| + |F|)$. □

Lemma 60 (Scaled triangle)**.** *Let $r > 0$ and $N, F \in \mathbb{C}$. We have $r|N - F| \le r|N| + r|F|$*

Proof. Apply Lemma 59 □

Lemma 61 (Ineq step)**.** *Let $0 < r < R$ and $N, F \in \mathbb{C}$. If $R|F| \le r|N-F|$ then $R|F| \le r|N|+r|F|$*

Proof. Apply assumption $R|F| \le r|N - F|$ and Lemma 60 □

Lemma 62 (Rearranged bound)**.** *Let $0 < r < R$ and $N, F \in \mathbb{C}$. If $R|F| \le r|N - F|$ then $(R - r)|F| \le r|N|$*

Proof. Apply Lemma 61 □

Lemma 63 (Abs positive)**.** *For $a \in \mathbb{R}$, if $a > 0$ then $|a| = a$.*

Proof. □

Lemma 64 (Double positive)**.** *For $a \in \mathbb{R}$, if $a > 0$ then $2a > 0$.*

Proof. □

Lemma 65 (Scaled abs)**.** *For $a \in \mathbb{R}$, if $a > 0$ then $|2a| = 2a$.*

Proof. Apply Lemmas 63 and 64 □

Lemma 66 (Key bound)**.** *Let $0 < r < R$, $M > 0$, and $F \in \mathbb{C}$. If $RF \le r|2M - F|$ then $(R - r)|F| \le 2Mr$*

Proof. Apply Lemma 62 with $N = 2M$, and Lemma 65 with $a = M$. □

Lemma 67 (Nonneg factor)**.** *Let $0 < r < R$ and $F \in \mathbb{C}$. Then we have $(R - r)|F| \ge 0$*

Proof. □

Lemma 68 (Divide bound)**.** *Let $0 < r < R$, $M > 0$, and $F \in \mathbb{C}$. If $(R - r)|F| \le 2Mr$ then $|F| \le \frac{2Mr}{R-r}$.*

Proof. Apply Lemma 52 with $c = (R - r) > 0$ and $a = (R - r)|F|$ and $b = 2Mr$. Lemma 67 gives $a \geq 0$. □

Lemma 69 (Final bound)**.** *Let $0 < r < R$, $M > 0$, and $F \in \mathbb{C}$. If $R|F| \leq r|2M - F|$ then $|F| \leq \frac{2Mr}{R-r}$*

Proof. Apply Lemmas 66 and 68. □

Lemma 70 (Order nonzero)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic, and let n_0 be the analyticOrderAt for f at 0. If $f(0) = 0$ then $n_0 \neq 0$.*

Proof. □

Lemma 71 (Order natural)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic, and let n_0 be the analyticOrderAt for f at 0. If $f \neq 0$ then $n_0 \in \mathbb{N}$.*

Proof. □

Lemma 72 (Factor power)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic at 0, and let n_0 be the analyticOrderAt for f at 0. If $n_0 \in \mathbb{N}$ then there exists a nhd N of 0 and $g : \mathbb{C} \to \mathbb{C}$ such that g is analytic at 0, and $f(z) = z^{n_0} g(z)$ on N.*

Proof. □

Lemma 73 (Factor linear)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic at 0, and let n_0 be the analyticOrderAt for f at 0. If $n_0 \in \mathbb{N}$ and $n_0 \neq 0$, then there exists a nhd N of 0 and $h : \mathbb{C} \to \mathbb{C}$ such that h is analytic at 0, and $f(z) = zh(z)$ on N.*

Proof. Apply Lemma 72 and let $h(z) = z^{n_0-1} g(z)$. □

Lemma 74 (Divide zero)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic at 0, and let n_0 be the analyticOrderAt for f at 0. If $f \neq 0$ and $f(0) = 0$, then $h(z) = f(z)/z$ is analytic at 0.*

Proof. Apply Lemmas 71 and 73 and 70 □

Lemma 75 (Inverse analytic)**.** *The function $f_1(z) = \frac{1}{z}$ is analytic on $\{z \in \mathbb{C} : z \neq 0\}$.*

Proof. □

Lemma 76 (Analytic mono)**.** *Let $T \subset S \subset \mathbb{C}$ and $f : S \to \mathbb{C}$. If f is analytic on S then f is analytic on T.*

Proof. □

Lemma 77 (Nonzero subset)**.** *Let $0 < R < 1$ and $V = \{z \in \overline{\mathbb{D}}_R : z \neq 0\}$ and $U = \{z \in \mathbb{C} : z \neq 0\}$. Then $V \subset U$.*

Proof. unfold definitions, using $\overline{\mathbb{D}}_R \subset \mathbb{C}$. □

Lemma 78 (Inverse analytic)**.** *The function $f_1(z) = \frac{1}{z}$ is analytic on $T = \{z \in \overline{\mathbb{D}}_R : z \neq 0\}$.*

Proof. Apply Lemmas 77 and 75 and 76 with $S = \{z \in \mathbb{C} : z \neq 0\}$ and $T = \{z \in \overline{\mathbb{D}}_R : z \neq 0\}$. □

Lemma 79 (Product analytic)**.** *Let $T \subset S \subset \mathbb{C}$, and let $f_1 : S \to \mathbb{C}$ and $f_2 : S \to \mathbb{C}$. If f_1 is analytic on T and f_2 is analytic on T, then $f_1 \cdot f_2$ is analytic on T.*

Proof. □

Lemma 80 (Product analytic). *Let $T \subset S \subset \mathbb{C}$, and let $f_1 : S \to \mathbb{C}$ and $f_2 : S \to \mathbb{C}$. If f_1 is analytic on T and f_2 is analytic on S, then $f_1 \cdot f_2$ is analytic on T.*

Proof. Apply Lemmas 79 and 76 with $f = f_2$ □

Lemma 81 (Product analytic). *Let $T = \{z \in \overline{\mathbb{D}}_R : z \neq 0\}$ and $f_1 : \mathbb{C} \to \mathbb{C}$ and $f_2 : \mathbb{C} \to \mathbb{C}$. If f_1 is analytic on T and f_2 is analytic on $\overline{\mathbb{D}}_R$, then $f_1 \cdot f_2$ is analytic on T.*

Proof. Apply Lemmas 80 with $S = \overline{\mathbb{D}}_R$ and $T = \{z \in \overline{\mathbb{D}}_R : z \neq 0\}$. □

Lemma 82 (Quotient analytic). *Let $T = \{z \in \overline{\mathbb{D}}_R : z \neq 0\}$ and $f : \mathbb{C} \to \mathbb{C}$. If $f(z)$ is analytic on $\overline{\mathbb{D}}_R$, then $f(z)/z$ is analytic on T.*

Proof. Apply Lemmas 78 and 81 with $f_1(z) = 1/z$ and $f_2(z) = f(z)$. □

Lemma 83 (On implies within). *Let $V \subset \mathbb{C}$ and $h : \mathbb{C} \to \mathbb{C}$. If h is AnalyticOn V, then h is AnalyticWithinAt z for all $z \in V$.*

Proof. □

Lemma 84 (Within implies on). *Let $h : \mathbb{C} \to \mathbb{C}$. If h is AnalyticWithinAt z for all $z \in \overline{\mathbb{D}}_R$, then h is AnalyticOn $\overline{\mathbb{D}}_R$.*

Proof. □

Lemma 85 (Disk split). *Let $T = \{z \in \overline{\mathbb{D}}_R : z \neq 0\}$. Then $\overline{\mathbb{D}}_R = \{0\} \cup T$.*

Proof. Unfold definition of T. □

Lemma 86 (Within union). *Let $T = \{z \in \overline{\mathbb{D}}_R : z \neq 0\}$ and $h : \mathbb{C} \to \mathbb{C}$. If h is AnalyticWithinAt 0 and h is AnalyticWithinAt z for all $z \in T$, then h is AnalyticWithinAt z for all $z \in \overline{\mathbb{D}}_R$.*

Proof. Apply Lemma 85. □

Lemma 87 (Within gives on). *Let $T = \{z \in \overline{\mathbb{D}}_R : z \neq 0\}$ and $h : \mathbb{C} \to \mathbb{C}$. If h is AnalyticWithinAt 0 and h is AnalyticWithinAt z for all $z \in T$, then h is AnalyticOn $\overline{\mathbb{D}}_R$.*

Proof. Apply Lemmas 84 and 86 □

Lemma 88 (At to within). *Let $h : \mathbb{C} \to \mathbb{C}$. If h is AnalyticAt 0, then h is AnalyticWithinAt 0.*

Proof. □

Lemma 89 (Local to global). *Let $T = \{z \in \overline{\mathbb{D}}_R : z \neq 0\}$ and $h : \mathbb{C} \to \mathbb{C}$. If h is AnalyticAt 0 and h is AnalyticOn T, then h is AnalyticOn $\overline{\mathbb{D}}_R$.*

Proof. Apply Lemmas 88 and 87. □

1.1 Borel-Carathéodory I

Definition 90 (Open disk)**.** For $R > 0$, define the open ball $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$.

Lemma 91 (Disk closure)**.** *For $R > 0$, the closure of $\mathbb{D}_R$ equals $\overline{\mathbb{D}}_R := \{z \in \mathbb{C} : |z| \leq R\}$*

Proof. □

Lemma 92 (In disk bound)**.** *For $R > 0$, if $w \in \overline{\mathbb{D}}_R$ then $|w| \leq R$.*

Proof. Apply Lemma 91. □

Lemma 93 (Outside disk bound)**.** *For $R > 0$, if $w \notin \mathbb{D}_R$ then $|w| \geq R$.*

Proof. Apply definition 90. □

Lemma 94 (Modulus equal)**.** *For $R > 0$, if $|w| \leq R$ and $|w| \geq R$ then $|w| = R$.*

Proof. □

Lemma 95 (Boundary modulus)**.** *For $R > 0$, if $w \in \overline{\mathbb{D}}_R$ and $w \notin \mathbb{D}_R$, then $|w| = R$.*

Proof. Apply Lemmas 92 and 93 and 94. □

Lemma 96 (Positive modulus)**.** *For $R > 0$ we have $|R| = R$.*

Proof. □

Lemma 97 (Modulus bound)**.** *For $R > 0$ we have $|R| \leq R$.*

Proof. Apply Lemma 96 and $R \leq R$. □

Lemma 98 (Positive radius belongs to its closed disk)**.** *For $R > 0$ we have $R \in \overline{\mathbb{D}}_R$.*

Proof. Apply Lemma 97 and definition 90 □

Lemma 99 (Compactness)**.** *For $R > 0$ the ball $\overline{\mathbb{D}}_R$ is a compact subset of $\mathbb{C}$.*

Proof. □

Lemma 100 (ExtrValThm)**.** *If $K \subset \mathbb{C}$ is compact and $g : K \to \mathbb{C}$ is continuous, then there exists $v \in K$ such that $|g(v)| \geq |g(z)|$ for all $z \in K$.*

Proof. □

Lemma 101 (Disk Boundary)**.** *If $g : \overline{\mathbb{D}}_R \to \mathbb{C}$ is continuous, then there exists $v \in \overline{\mathbb{D}}_R$ such that $|g(v)| \geq |g(z)|$ for all $z \in \overline{\mathbb{D}}_R$.*

Proof. Apply Lemmas 99 and 100 with $K = \overline{\mathbb{D}}_R$. □

Lemma 102 (Analytic Continuation)**.** *If $h : \overline{\mathbb{D}}_R \to \mathbb{C}$ is analytic, then h is continuous.*

Proof. □

Lemma 103 (Max modulus)**.** *If $h : \overline{\mathbb{D}}_R \to \mathbb{C}$ is analytic, then there exists $u \in \overline{\mathbb{D}}_R$ such that $|h(u)| \geq |h(z)|$ for all $z \in \overline{\mathbb{D}}_R$.*

Proof. Apply Lemmas 101 and 102 with $g(z) = h(z)$ and $u = v$. □

Lemma 104 (Interior max)**.** *Let* $R > 0$*. Let* $h : \overline{\mathbb{D}}_R \to \mathbb{C}$ *be analytic. Suppose there exists* $w \in \mathbb{D}_R$ *such that* $|h(w)| \geq |h(z)|$ *for all* $z \in \mathbb{D}_R$*. Then* $|h(z)| = |h(w)|$ *for all* $z \in \overline{\mathbb{D}}_R$*.*

Proof. Apply Lemma 91 ☐

Lemma 105 (Boundary value)**.** *Let* $R > 0$*. Let* $h : \overline{\mathbb{D}}_R \to \mathbb{C}$ *be analytic. Suppose there exists* $w \in \mathbb{D}_R$ *such that* $|h(w)| \geq |h(z)|$ *for all* $z \in \mathbb{D}_R$*. Then* $|h(R)| = |h(w)|$*.*

Proof. Apply Lemmas 104 and 98 with $z = R$. ☐

Lemma 106 (Boundary bound)**.** *Let* $R > 0$*. Let* $h : \overline{\mathbb{D}}_R \to \mathbb{C}$ *be analytic. Suppose there exists* $w \in \mathbb{D}_R$ *such that* $|h(w)| \geq |h(z)|$ *for all* $z \in \mathbb{D}_R$*. Then* $|h(R)| \geq |h(z)|$ *for all* $z \in \overline{\mathbb{D}}_R$

Proof. Apply Lemmas 104 and 105 ☐

Lemma 107 (Boundary point)**.** *Let* $R > 0$*. Let* $h : \overline{\mathbb{D}}_R \to \mathbb{C}$ *be analytic. There exists* $v \in \overline{\mathbb{D}}_R$ *with* $|v| = R$ *such that* $|h(v)| \geq |h(z)|$ *for all* $z \in \overline{\mathbb{D}}_R$

Proof. Apply Lemma 103 to get $u \in \overline{\mathbb{D}}_R$ such that $|h(u)| \geq |h(z)|$ for all $z \in \overline{\mathbb{D}}_R$. If $u \in \mathbb{D}_R$ then set $v = R$. Now by Lemma 106 we have $|v| = |R| = R$. Else $u \notin \mathbb{D}_R$, then set $v = u$. Now by Lemma 95 with $w = u$ we have $|v| = |u| = R$. ☐

Lemma 108 (Boundary point)**.** *Let* $R > 0$*. Let* $h(z)$ *be analytic on* $|z| \leq R$*. There exists* $v \in \mathbb{C}$ *with* $|v| = R$ *such that* $|h(v)| \geq |h(z)|$ *for all* $z \in \mathbb{C}$ *with* $|z| \leq R$*.*

Proof. Apply Lemma 107, and apply Lemma 92 for $w = v$ and then again for $w = z$. ☐

Lemma 109 (Boundary max)**.** *Let* $R > 0$ *and* $B \geq 0$*. Let* $h(z)$ *be a function analytic on* $|z| \leq R$*. Suppose that* $|h(z)| \leq B$ *for all* $z \in \mathbb{C}$ *with* $|z| = R$*. Then there exists* $v \in \mathbb{C}$ *with* $|v| = R$ *such that* $|h(v)| \geq |h(w)|$ *for all* $w \in \mathbb{C}$ *with* $|w| \leq R$*, and* $|h(v)| \leq B$*.*

Proof. Apply Lemma 108, and the assumption $|h(z)| \leq B$ with $z = v$, since $|v| = R$. ☐

Lemma 110 (Max principle)**.** *Let* $R > 0$ *and* $B \geq 0$*. Let* $h(z)$ *be a function analytic on* $|z| \leq R$*. Suppose that* $|h(z)| \leq B$ *for all* $z \in \mathbb{C}$ *with* $|z| = R$*. Then* $|h(w)| \leq B$ *for all* $w \in \mathbb{C}$ *with* $|w| \leq R$*.*

Proof. Apply Lemma 109. By assumption we calculate $|h(w)| \leq |h(v)| \leq B$ for all $w \in \mathbb{C}$ with $|w| \leq R$. ☐

Lemma 111 (Easy maximum principle)**.** *Let* $R > 0$ *and* $B \geq 0$*. Let* $h(z)$ *be a function analytic on* $|z| \leq R$*. Suppose that* $|h(w)| \leq B$ *for all* $w \in \mathbb{C}$ *with* $|w| \leq R$*. Then* $|h(z)| \leq B$ *for all* $z \in \mathbb{C}$ *with* $|z| = R$*.*

Proof. Take $z \in \mathbb{C}$ with $|z| = R$. Then $|z| \leq R$, so by assumption with $w = z$ we have $|h(z)| \leq B$. ☐

Lemma 112 (Maximum modulus principle)**.** *Let* $T > 0$ *and* $B \geq 0$*. Let* $h(z)$ *be a function analytic on* $|z| \leq T$*. We have* $|h(z)| \leq B$ *for all* $z \in \mathbb{C}$ *with* $|z| \leq T$ *if and only if* $|h(z)| \leq B$ *for all* $z \in \mathbb{C}$ *with* $|z| = T$*.*

Proof. Apply Lemmas 110 and 111 with $R = T$. ☐

Lemma 113 (Denom nonzero)**.** *Let* $R, M > 0$*. Suppose* $f(z)$ *is an analytic function on* $|z| \leq R$ *satisfying* $\Re(f(z)) \leq M$*. Then* $2M - f(z) \neq 0$ *for all* $|z| \leq R$*.*

Proof. For each z with $|z| \leq R$, apply Lemma 34 with $w = f(z)$. ☐

Lemma 114 (Ratio bound)**.** *Let $R, M > 0$. Suppose $f(z)$ is an analytic function on $|z| \le R$ satisfying $\Re(f(z)) \le M$. For any z with $|z| \le R$, we have $\frac{|f(z)|}{|2M - f(z)|} \le 1$.*

Proof. For each z with $|z| \le R$, apply Lemma 56 with $w = f(z)$. □

Lemma 115 (Removable zero)**.** *Let $R > 0$. Let f be analytic on $|z| \le R$ such that $f(0) = 0$. Then the function $h(z) = f(z)/z$ is analytic on $|z| \le R$.*

Proof. Apply theorems 74, 82 and 89. □

Lemma 116 (Quotient analytic)**.** *Let $R > 0$. If $h_1(z)$ and $h_2(z)$ are analytic for $|z| \le R$ and $h_2(z) \ne 0$ for all $|z| \le R$, then $h_1(z)/h_2(z)$ is analytic for $|z| \le R$.*

Proof. □

Definition 117 (Modified function)**.** Let $R, M > 0$. Let f be analytic on $|z| \le R$ such that $f(0) = 0$ and suppose $\Re f(z) \le M$ for all $|z| \le R$. Define the function $f_M(z)$ for $|z| \le R$ as

$$f_M(z) = \frac{f(z)/z}{2M - f(z)}.$$

Lemma 118 (g analytic)**.** *The function $f_M(z)$ from Definition 117 is analytic on $|z| \le R$.*

Proof. Write $f_M(z) = h_1(z)/h_2(z)$ where $h_1(z) = f(z)/z$ and $h_2(z) = 2M - f(z)$. Then apply Lemma 116 with $h_1(z)$ and $h_2(z)$, using Lemma 115 and Lemma 113. □

Lemma 119 (Quotient modulus)**.** *Let $a, b \in \mathbb{C}$. If $b \ne 0$ then $|a/b| = |a|/|b|$.*

Proof. □

Lemma 120 (g modulus)**.** *Let $R, M > 0$. Let f be analytic on $|z| \le R$ such that $f(0) = 0$ and suppose $\Re f(z) \le M$ for all $|z| \le R$. We have $|f_M(z)| = \frac{|f(z)/z|}{|2M - f(z)|}$.*

Proof. For each $|z| \le R$, apply Definition 117, and Lemma 119 with $a = f(z)/z$ and $b = 2M - f(z)$. Note $b \ne 0$ by Lemma 113. □

Lemma 121 (Quotient radius)**.** *Let $T > 0$ and $z, w \in \mathbb{C}$. If $|z| = T$ then $|w/z| = |w|/T$.*

Proof. □

Lemma 122 (Boundary g)**.** *Let $R, M > 0$. Let f be analytic on $|z| \le R$ such that $f(0) = 0$ and suppose $\Re f(z) \le M$ for all $|z| \le R$. For any $z \in \mathbb{C}$ with $|z| = R$, we have $|f_M(z)| = \frac{|f(z)|/R}{|2M - f(z)|}$.*

Proof. Apply Lemmas 120 and 121 with $w = f(z)$ and $T = R > 0$. □

Lemma 123 (Scaled ratio)**.** *Let $R, M > 0$. Let f be analytic on $|z| \le R$ such that $f(0) = 0$ and suppose $\Re f(z) \le M$ for all $|z| \le R$. For any z with $|z| \le R$, we have $\frac{|f(z)|/R}{|2M - f(z)|} \le 1/R$.*

Proof. Apply Lemma 114. □

Lemma 124 (Boundary bound)**.** *Let $R, M > 0$. Let f be analytic on $|z| \le R$ such that $f(0) = 0$ and suppose $\Re f(z) \le M$ for all $|z| \le R$. For any $z \in \mathbb{C}$ with $|z| = R$, we have $|f_M(z)| \le 1/R$.*

Proof. Apply Lemmas 122 and 123. □

Lemma 125 (Interior bound)**.** *Let $R, M > 0$. Let f be analytic on $|z| \leq R$ such that $f(0) = 0$ and suppose $\Re f(z) \leq M$ for all $|z| \leq R$. For any $z \in \mathbb{C}$ with $|z| \leq R$, we have $|f_M(z)| \leq 1/R$.*

Proof. Apply Lemmas 124 and 112 with $B = 1/R$ and $T = R$ and $h(z) = f_M(z)$. □

Lemma 126 (g at r)**.** *Let $R, M > 0$. Let f be analytic on $|z| \leq R$ such that $f(0) = 0$ and suppose $\Re f(z) \leq M$ for all $|z| \leq R$. For any $0 < r < R$ and any $z \in \mathbb{C}$ with $|z| = r$, we have $|f_M(z)| = \frac{|f(z)|/r}{|2M - f(z)|}$.*

Proof. Apply Lemmas 120 and 121 with $w = f(z)$ and $T = r > 0$. □

Lemma 127 (g bound r)**.** *Let $R, M > 0$. Let f be analytic on $|z| \leq R$ such that $f(0) = 0$ and suppose $\Re f(z) \leq M$ for all $|z| \leq R$. For any $0 < r < R$ and any $z \in \mathbb{C}$ with $|z| = r$, we have $\frac{|f(z)|/r}{|2M - f(z)|} \leq 1/R$.*

Proof. Apply Lemmas 125 and 126 with $|z| = r < R$. □

Lemma 128 (Fraction swap)**.** *Let $a, b, r, R > 0$. If $\frac{a/r}{b} \leq 1/R$ then $Ra \leq rb$.*

Proof. □

Lemma 129 (Rearranged bound)**.** *Let $R, M > 0$. Let f be analytic on $|z| \leq R$ such that $f(0) = 0$ and suppose $\Re f(z) \leq M$ for all $|z| \leq R$. For any $0 < r < R$ and any $z \in \mathbb{C}$ with $|z| = r$, we have $R|f(z)| \leq r|2M - f(z)|$.*

Proof. Apply Lemmas 127 and 128 with $a = |f(z)| > 0$ and $b = |2M - f(z)| > 0$. □

Lemma 130 (Circle bound)**.** *Let $R, M > 0$. Let f be analytic on $|z| \leq R$ such that $f(0) = 0$ and suppose $\Re f(z) \leq M$ for all $|z| \leq R$. For any $0 < r < R$ and any $z \in \mathbb{C}$ with $|z| = r$, we have*

$$|f(z)| \leq \frac{2r}{R - r} M.$$

Proof. For each $|z| \leq R$, apply Lemmas 129 and 69 with $F = f(z)$. □

Lemma 131 (Circle bound)**.** *Let $R, M > 0$. Let f be analytic on $|z| \leq R$ such that $f(0) = 0$ and suppose $\Re f(z) \leq M$ for all $|z| \leq R$. For any $0 < r < R$, and any $z \in \mathbb{C}$ with $|z| = r$ we have*

$$|f(z)| \leq \frac{2r}{R - r} M.$$

Proof. Apply Lemma 130. □

Lemma 132 (BC bound)**.** *Let $R, M > 0$. Let f be analytic on $|z| \leq R$ such that $f(0) = 0$ and suppose $\Re f(z) \leq M$ for all $|z| \leq R$. For any $0 < r < R$, and any $z \in \mathbb{C}$ with $|z| \leq r$ we have*

$$|f(z)| \leq \frac{2r}{R - r} M.$$

Proof. Apply Lemmas 131 and 112 with $B = \frac{2r}{R-r} M$ and $T = r$ and $h(z) = f(z)$. □

Theorem 133 (Borel-Carathéodory I)**.** *Let $R, M > 0$. Let f be analytic on $|z| \leq R$ such that $f(0) = 0$ and suppose $\Re f(z) \leq M$ for all $|z| \leq R$. For any $0 < r < R$,*

$$\sup_{|z| \leq r} |f(z)| \leq \frac{2r}{R - r} M.$$

Proof. Apply Lemma 132 and definition of supremum $\sup_{|z| \leq r}$. □

1.2 Borel-Carathéodory II

Lemma 134 (Cauchy's Integral Formula for f')**.** *Let f be analytic on $|z| \leq R$. For any z with $|z| \leq r$ and any r' with $0 < r < r' < R$,*

$$f'(z) = \frac{1}{2\pi i} \oint_{|w|=r'} \frac{f(w)}{(w-z)^2} dw.$$

Proof. □

Lemma 135 (Differential of $w(t)$)**.** *For $w(t) = r'e^{it}$, we have $dw = ir'e^{it}dt$.*

Proof. Differentiate $w(t)$ with respect to t. □

Lemma 136 (CIF for f', Parameterized)**.** *Let f be analytic on $|z| \leq R$. For any z with $|z| \leq r$ and any r' with $0 < r < r' < R$,*

$$f'(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(r'e^{it})}{(r'e^{it} - z)^2} (ir'e^{it}) \, dt.$$

Proof. Apply Lemmas 134 and 135, and unfold definition of the circle integral $\oint$ over $w \in C(0, r')$. □

Lemma 137 (CIF for f', Simplified)**.** *Let f be analytic on $|z| \leq R$. For any z with $|z| \leq r$ and any r' with $0 < r < r' < R$,*

$$f'(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(r'e^{it})r'e^{it}}{(r'e^{it} - z)^2} \, dt.$$

Proof. Apply Lemma 136 and cancel i from the numerator and denominator. □

Lemma 138 (Derivative modulus)**.** *Let $0 < r < r' < R$. Let f be analytic on $|z| \leq R$. For any z with $|z| \leq r$, we have $|f'(z)| = \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{f(r'e^{it})r'e^{it}}{(r'e^{it}-z)^2} \, dt \right|$.*

Proof. Apply modulus to both sides of the equality in Lemma 137. □

Lemma 139 (Integral bound)**.** *For an integrable function $g(t)$, we have $|\int_a^b g(t)dt| \leq \int_a^b |g(t)|dt$.*

Proof. □

Lemma 140 (Modulus of f')**.** *Let $0 < r < r' < R$. Let f be analytic on $|z| \leq R$. For any z with $|z| \leq r$, we have $|f'(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(r'e^{it})r'e^{it}}{(r'e^{it}-z)^2} \right| \, dt$.*

Proof. Apply Lemmas 138 and 139. □

Lemma 141 (Integrand modulus)**.** *Let $0 < r < r' < R$. Let f be analytic on $|z| \leq R$. For any z with $|z| \leq r$, we have $|f(r'e^{it})r'e^{it}| = |f(r'e^{it})| \cdot |r'e^{it}|$.*

Proof. Apply modulus property $|ab| = |a||b|$. □

Lemma 142 (Modulus one)**.** *For $t \in \mathbb{R}$ we have $|e^{it}| = e^{\Re(it)}$*

Proof. □

Lemma 143 (Cosine part)**.** *For* $t \in \mathbb{R}$ *we have* $\Re(it) = 0$

Proof. □

Lemma 144 (Euler part)**.** *For* $t \in \mathbb{R}$ *we have* $e^{\Re(it)} = e^0$*.*

Proof. Apply Lemma 143. □

Lemma 145 (Exp zero one)**.** *We have* $e^0 = 1$*.*

Proof. □

Lemma 146 (Cosine relation)**.** *For* $t \in \mathbb{R}$ *we have* $e^{\Re(it)} = 1$

Proof. Apply Lemmas 144 and 145 □

Lemma 147 (Unit modulus)**.** *For* $t \in \mathbb{R}$ *we have* $|e^{it}| = 1$

Proof. Apply Lemmas 142 and 146 □

Lemma 148 (Scaled modulus)**.** *For* $a > 0$ *and* $t \in \mathbb{R}$ *we have* $|ae^{it}| = a$

Proof. Apply Lemma 147 and calculate $|ae^{it}| = |a| \cdot |e^{it}| = a \cdot 1 = a$. □

Lemma 149 (Integrand modulus)**.** *Let* $0 < r < r' < R$*. Let* f *be analytic on* $|z| \leq R$*. For any* z *with* $|z| \leq r$*, we have* $|f(r'e^{it})r'e^{it}| = r'|f(r'e^{it})|$*.*

Proof. Apply Lemmas 141 and 148 with $a = r'$. □

Lemma 150 (Square modulus)**.** *For any* $c \in \mathbb{C}$*,* $|c^2| = |c|^2$*.*

Proof. □

Lemma 151 (Shifted modulus)**.** *For any* $w, z \in \mathbb{C}$*,* $|(w - z)^2| = |w - z|^2$*.*

Proof. Apply Lemma 150 with $c = w - z$. □

Lemma 152 (Reverse triangle)**.** *For any* $w, z \in \mathbb{C}$*, we have* $|w| - |z| \leq |w - z|$*.*

Proof. □

Lemma 153 (Reverse triangle)**.** *Let* $t \in \mathbb{R}$ *and* $0 < r < r' < R$ *and* $z \in \mathbb{C}$*, we have* $|r'e^{it}| - |z| \leq |r'e^{it} - z|$*.*

Proof. Apply Lemma 152 with $w = r'e^{it}$. □

Lemma 154 (Reverse triangle)**.** *Let* $t \in \mathbb{R}$ *and* $0 < r < r' < R$ *and* $z \in \mathbb{C}$*, we have* $r' - |z| \leq |r'e^{it} - z|$*.*

Proof. Apply Lemmas 153 and 148 with $a = r'$ □

Lemma 155 (Radius relation)**.** *Let* $0 < r < r' < R$ *and* $z \in \mathbb{C}$ *with* $|z| \leq r$*. Then* $0 < r' - |z|$*.*

Proof. We calculate $|z| \leq r < r'$ by assumption, so $0 < r' - |z|$. □

Lemma 156 (Radius relation)**.** *Let* $t \in \mathbb{R}$ *and* $0 < r < r' < R$ *and* $z \in \mathbb{C}$ *and* $z \in \mathbb{C}$ *with* $|z| \leq r$*. Then* $r' - r \leq |r'e^{it} - z|$*.*

Proof. Apply Lemma 154 and $|z| \leq r$. □

Lemma 157 (Radius relation)**.** *If* $0 < r < r'$ *then* $r' - r > 0$

Proof. Calculation □

Lemma 158 (Radius relation)**.** *If* $0 < r < r'$ *then* $(r' - r)^2 > 0$

Proof. Apply Lemma 157 □

Lemma 159 (Radius relation)**.** *Let* $t \in \mathbb{R}$ *and* $0 < r < r' < R$ *and* $z \in \mathbb{C}$*and* $z \in \mathbb{C}$ *with* $|z| \leq r$*. Then* $(r' - r)^2 \leq |r'e^{it} - z|^2$*.*

Proof. Apply Lemmas 156 and 158. □

Lemma 160 (Radius relation)**.** *Let* $t \in \mathbb{R}$ *and* $0 < r < r' < R$ *and* $z \in \mathbb{C}$*and* $z \in \mathbb{C}$ *with* $|z| \leq r$*. Then* $|(r'e^{it} - z)^2| = |r'e^{it} - z|^2$*.*

Proof. Apply Lemmas 156 and 158. □

Lemma 161 (Reverse triangle)**.** *Let* $t \in \mathbb{R}$ *and* $0 < r < r' < R$ *and* $z \in \mathbb{C}$ *with* $|z| \leq r$*, we have* $0 < |r'e^{it} - z|$*.*

Proof. Apply Lemmas 154 and 155. □

Lemma 162 (Positive nonzero)**.** *For* $w \in \mathbb{C}$*, if* $|w| > 0$ *then* $w \neq 0$*.*

Proof. □

Lemma 163 (Reverse triangle)**.** *Let* $t \in \mathbb{R}$ *and* $0 < r < r' < R$ *and* $z \in \mathbb{C}$ *with* $|z| \leq r$*, we have* $r'e^{it} - z \neq 0$*.*

Proof. Apply Lemmas 161 and 162 with $w = r'e^{it} - z$. □

Lemma 164 (Reverse triangle)**.** *Let* $t \in \mathbb{R}$ *and* $0 < r < r' < R$ *and* $z \in \mathbb{C}$ *with* $|z| \leq r$*, we have* $(r'e^{it} - z)^2 \neq 0$*.*

Proof. Apply Lemma 163, and Mathlib mul_self_ne_zero □

Lemma 165 (Division bound)**.** *If* $a, b \in \mathbb{C}$ *and* $b \neq 0$ *then* $|a/b| = |a|/|b|$*.*

Proof. □

Lemma 166 (Integrand modulus)**.** *Let* $0 < r < r' < R$*. Let* f *be analytic on* $|z| \leq R$*. For any* z *with* $|z| \leq r$*, we have* $\left|\frac{f(r'e^{it})r'e^{it}}{(r'e^{it}-z)^2}\right| = \frac{|f(r'e^{it})r'e^{it}|}{|(r'e^{it}-z)^2|}$*.*

Proof. Apply Lemma 165 with $a = f(r'e^{it})r'e^{it}$ and $b = (r'e^{it} - z)^2$. Here $b \neq 0$ by Lemma 164. □

Lemma 167 (Product modulus)**.** *Let* $0 < r < r' < R$*. Let* f *be analytic on* $|z| \leq R$*. For any* z *with* $|z| \leq r$*, we have* $\left|\frac{f(r'e^{it})r'e^{it}}{(r'e^{it}-z)^2}\right| = \frac{r'|f(r'e^{it})|}{|(r'e^{it}-z)^2|}$*.*

Proof. Apply Lemmas 166 and 149. □

Lemma 168 (Product modulus)**.** *Let* $0 < r < r' < R$*. Let* f *be analytic on* $|z| \leq R$*. For any* z *with* $|z| \leq r$*, we have* $\left|\frac{f(r'e^{it})r'e^{it}}{(r'e^{it}-z)^2}\right| = \frac{r'|f(r'e^{it})|}{|r'e^{it}-z|^2}$*.*

Proof. Apply Lemmas 167 and 160. □

Lemma 169 (Product modulus). *Let $0 < r < r' < R$. Let f be analytic on $|z| \leq R$. For any z with $|z| \leq r$, we have $\frac{r'|f(r'e^{it})|}{|r'e^{it}-z|^2} \leq \frac{r'|f(r'e^{it})|}{(r'-r)^2}$.*

Proof. Apply Lemmas 168 and 159. ☐

Lemma 170 (Product modulus). *Let $0 < r < r' < R$. Let f be analytic on $|z| \leq R$. For any z with $|z| \leq r$, we have $\left|\frac{f(r'e^{it})r'e^{it}}{(r'e^{it}-z)^2}\right| \leq \frac{r'|f(r'e^{it})|}{(r'-r)^2}$.*

Proof. Apply Lemmas 168 and 169. ☐

Lemma 171 (Point bound). *Let $M, R > 0$ and $0 < r' < R$. Let f be analytic on $|z| \leq R$ such that $f(0) = 0$ and suppose $\Re f(z) \leq M$ for all $|z| \leq R$. For any $t \in \mathbb{R}$ we have $|f(r'e^{it})| \leq \frac{2r'M}{R-r'}$.*

Proof. Note $w = r'e^{it}$ satisfies $|w| = r' < R$ by Lemma 148 with $a = r'$. Then apply Lemma 132. ☐

Lemma 172 (Integrand bound). *Let $M, R > 0$ and $0 < r < r' < R$. Let f be analytic on $|z| \leq R$ such that $f(0) = 0$ and suppose $\Re f(z) \leq M$ for all $|z| \leq R$. For any $t \in \mathbb{R}$ we have $\left|\frac{f(r'e^{it})r'e^{it}}{(r'e^{it}-z)^2}\right| \leq \frac{2(r')^2M}{(R-r')(r'-r)^2}$.*

Proof. Apply Lemmas 170 and 171. ☐

Lemma 173 (Integral inequality). *If $g(t) \leq C$ for all $t \in [a, b]$, then $\int_a^b g(t)dt \leq \int_a^b Cdt$.*

Proof. ☐

Lemma 174 (Derivative bound). *Let $M, R > 0$ and $0 < r < r' < R$. Let f be analytic on $|z| \leq R$ such that $f(0) = 0$ and suppose $\Re f(z) \leq M$ for all $|z| \leq R$. For any $t \in \mathbb{R}$ we have $|f'(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{2(r')^2M}{(R-r')(r'-r)^2}dt$.*

Proof. Apply Lemmas 140, 172 and 173 with $g(t) = \left|\frac{f(r'e^{it})r'e^{it}}{(r'e^{it}-z)^2}\right|$ and $C = \frac{2(r')^2M}{(R-r')(r'-r)^2}$ ☐

Lemma 175 (Integrate one). *We have $\int_0^{2\pi} dt = 2\pi$.*

Proof. ☐

Lemma 176 (Exponential integral). *We have $\frac{1}{2\pi}\int_0^{2\pi} dt = 1$.*

Proof. Apply Lemma 175 and simplify. ☐

Lemma 177 (Derivative bound). *Let $M, R > 0$ and $0 < r < r' < R$. Let f be analytic on $|z| \leq R$ such that $f(0) = 0$ and suppose $\Re f(z) \leq M$ for all $|z| \leq R$. Then we have $|f'(z)| \leq \frac{2(r')^2M}{(R-r')(r'-r)^2}$.*

Proof. Apply Lemmas 174 and 176. ☐

Lemma 178 (Radius compare). *Given $0 < r < R$ with $r' = \frac{r+R}{2}$, we have $r < r'$.*

Proof. Since $r < R$, we have $2r < r + R$. Dividing by 2 gives $r < (r + R)/2$. ☐

Lemma 179 (Radius compare). *Given $0 < r < R$ with $r' = \frac{r+R}{2}$, we have $r' < R$.*

Proof. Since $r < R$, we have $r + R < 2R$. Dividing by 2 gives $(r + R)/2 < R$. ☐

Lemma 180 (Intermediate radius). *Given $0 < r < R$ with $r' = \frac{r+R}{2}$, we have $r < r' < R$.*

Proof. Apply Lemmas 178 and 179. ☐

Lemma 181 (Radius formula)**.** *Given* $0 < r < R$ *with* $r' = \frac{r+R}{2}$*, we have* $R - r' = \frac{R-r}{2}$*.*

Proof. We calculate $R - \frac{r+R}{2} = \frac{2R-(r+R)}{2} = \frac{R-r}{2}$. ☐

Lemma 182 (Radius formula)**.** *Given* $0 < r < R$ *with* $r' = \frac{r+R}{2}$*, we have* $r' - r = \frac{R-r}{2}$*.*

Proof. We calculate $\frac{r+R}{2} - r = \frac{r+R-2r}{2} = \frac{R-r}{2}$. ☐

Lemma 183 (Denominator form)**.** *Given* $0 < r < R$ *with* $r' = \frac{r+R}{2}$*, we have* $(R-r')(r'-r)^2 = \frac{(R-r)^3}{8}$*.*

Proof. Apply Lemmas 181 and 182 and calculate $\left(\frac{R-r}{2}\right) \cdot \left(\frac{R-r}{2}\right)^2 = \frac{(R-r)}{2}\frac{(R-r)^2}{4} = \frac{(R-r)^3}{8}$. ☐

Lemma 184 (Numerator form)**.** *Given* $M > 0$ *and* $0 < r < R$ *with* $r' = \frac{r+R}{2}$*, we have* $2(r')^2M = \frac{(R+r)^2M}{2}$*.*

Proof. We calculate $2(r')^2M = 2\left(\frac{R+r}{2}\right)^2 M = 2\frac{(R+r)^2}{4}M = \frac{(R+r)^2M}{2}$ ☐

Lemma 185 (Fraction simplify)**.** *Given* $M > 0$ *and* $0 < r < R$ *with* $r' = \frac{r+R}{2}$*, we have* $\frac{2(r')^2M}{(R-r')(r'-r)^2} = \frac{(R+r)^2M/2}{(R-r)^3/8}$*.*

Proof. Apply Lemmas 184 and 183. ☐

Lemma 186 (Fraction simplify)**.** *Given* $M > 0$ *and* $0 < r < R$*, we have* $\frac{(R+r)^2M/2}{(R-r)^3/8} = \frac{4(R+r)^2M}{(R-r)^3}$*.*

Proof. Simplify fraction ☐

Lemma 187 (Fraction simplify)**.** *Given* $M > 0$ *and* $0 < r < R$ *with* $r' = \frac{r+R}{2}$*, we have* $\frac{2(r')^2M}{(R-r')(r'-r)^2} = \frac{4(R+r)^2M}{(R-r)^3}$*.*

Proof. Apply Lemmas 185 and 186. ☐

Lemma 188 (Inequality fact)**.** *Given* $r < R$*, we have* $R + r < 2R$*.*

Proof. calculation ☐

Lemma 189 (Sum positive)**.** *Given* $0 < r < R$*, we have* $0 < R + r$*.*

Proof. ☐

Lemma 190 (Double positive)**.** *Given* $0 < R$*, we have* $2R > 0$*.*

Proof. ☐

Lemma 191 (Square fact)**.** *If* $0 < a < b$*, then* $a^2 < b^2$*.*

Proof. ☐

Lemma 192 (Square bound)**.** *Given e, we have* $(R + r)^2 < (2R)^2$*.*

Proof. Let $a = R + r$ and $b = 2R$. From Lemma 189, $a > 0$. From Lemma 190, $b > 0$. From Lemma 188, $a < b$. Apply Lemma 191. ☐

Lemma 193 (Square identity). *For any $R > 0$, we have $(2R)^2 = 4R^2$.*

Proof. We calculate $(2R)^2 = 2^2R^2 = 4R^2$. □

Lemma 194 (Square bound). *Given $0 < r < R$, we have $(R + r)^2 < 4R^2$.*

Proof. Apply Lemmas 192 and 193. □

Lemma 195 (Square bound). *Given $M > 0$ and $0 < r < R$, we have $4(R + r)^2 M < 16R^2M$.*

Proof. Apply Lemma 194 and multiply by $4M > 0$. □

Lemma 196 (Bound simplify). *Given $M > 0$ and $0 < r < R$, we have $\frac{4(R+r)^2M}{(R-r)^3} < \frac{16R^2M}{(R-r)^3}$.*

Proof. Apply Lemma 195 to the numerator of the fraction. □

Lemma 197 (Fraction simplify). *Given $M > 0$ and $0 < r < R$ with $r' = \frac{r+R}{2}$, we have $\frac{2(r')^2M}{(R-r')(r'-r)^2} \leq \frac{16R^2M}{(R-r)^3}$.*

Proof. Apply Lemmas 187 and 196. □

Theorem 198 (Borel-Carathéodory II). *Let $R, M > 0$. Let f be analytic on $|z| \leq R$ such that $f(0) = 0$ and suppose $\Re f(z) \leq M$ for all $|z| \leq R$. For any $0 < r < R$ and any $|z| \leq r$,*

$$|f'(z)| \leq \frac{16MR^2}{(R-r)^3}.$$

Proof. Apply Lemmas 197 and 177 with $r' = \frac{r+R}{2}$. □

1.3 Integral Antiderivative

Lemma 199 (Cauchy rectangles). *Let $0 < R < R_0 < 1$, and assume $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ analyticOnNhd $\overline{\mathbb{D}}_{R_0}$. Then for any $z, w \in \overline{\mathbb{D}}_R$,*

$$\left(\int_{z.\mathrm{Re}}^{w.\mathrm{Re}} f(x + i\, z.\mathrm{Im})\, dx\right) - \left(\int_{z.\mathrm{Re}}^{w.\mathrm{Re}} f(x + i\, w.\mathrm{Im})\, dx\right) + i\left(\int_{z.\mathrm{Im}}^{w.\mathrm{Im}} f(w.\mathrm{Re} + iy)\, dy\right) - i\left(\int_{z.\mathrm{Im}}^{w.\mathrm{Im}} f(z.\mathrm{Re} + iy)\, dy\right) = 0.$$

Proof. Let the four corners of a rectangle be $A = z.\mathrm{Re} + i\, z.\mathrm{Im}$, $B = w.\mathrm{Re} + i\, z.\mathrm{Im}$, $C = w.\mathrm{Re} + i\, w.\mathrm{Im}$, and $D = z.\mathrm{Re} + i\, w.\mathrm{Im}$. Since $z, w \in \overline{\mathbb{D}}_R$, all four corners lie within the closed disk $\overline{\mathbb{D}}_{R_0}$. The assumption is that f is analytic on a neighborhood of $\overline{\mathbb{D}}_{R_0}$. This means there exists an open set U containing $\overline{\mathbb{D}}_{R_0}$ on which f is analytic. The rectangle with corners A, B, C, D is contained in $\overline{\mathbb{D}}_{R_0}$, and therefore also in U.

By Cauchy's Integral Theorem for a rectangle (Mathlib: integral_boundary_rect_eq_zero_of_differentiableOn), the integral of an analytic function over the boundary of the rectangle is zero. We can express this boundary integral as the sum of four path integrals:

$$\oint_{\partial \mathrm{Rect}} f(\zeta)\, d\zeta = \int_A^B f(\zeta)\, d\zeta + \int_B^C f(\zeta)\, d\zeta + \int_C^D f(\zeta)\, d\zeta + \int_D^A f(\zeta)\, d\zeta = 0.$$

We evaluate each integral:

1. Path from A to B: $\zeta(x) = x + i\, z.\mathrm{Im}$ for x from $z.\mathrm{Re}$ to $w.\mathrm{Re}$. So $d\zeta = dx$.

$$\int_A^B f(\zeta)\, d\zeta = \int_{z.\mathrm{Re}}^{w.\mathrm{Re}} f(x + i\, z.\mathrm{Im})\, dx.$$

2. Path from B to C: $\zeta(y) = w.\mathrm{Re} + i\, y$ for y from $z.\mathrm{Im}$ to $w.\mathrm{Im}$. So $d\zeta = i\, dy$.

$$\int_B^C f(\zeta)\, d\zeta = i \int_{z.\mathrm{Im}}^{w.\mathrm{Im}} f(w.\mathrm{Re} + iy)\, dy.$$

3. Path from C to D: $\zeta(x) = x + i\, w.\mathrm{Im}$ for x from $w.\mathrm{Re}$ to $z.\mathrm{Re}$. So $d\zeta = dx$.

$$\int_C^D f(\zeta)\, d\zeta = \int_{w.\mathrm{Re}}^{z.\mathrm{Re}} f(x + i\, w.\mathrm{Im})\, dx = -\int_{z.\mathrm{Re}}^{w.\mathrm{Re}} f(x + i\, w.\mathrm{Im})\, dx.$$

4. Path from D to A: $\zeta(y) = z.\mathrm{Re} + i\, y$ for y from $w.\mathrm{Im}$ to $z.\mathrm{Im}$. So $d\zeta = i\, dy$.

$$\int_D^A f(\zeta)\, d\zeta = i \int_{w.\mathrm{Im}}^{z.\mathrm{Im}} f(z.\mathrm{Re} + iy)\, dy = -i \int_{z.\mathrm{Im}}^{w.\mathrm{Im}} f(z.\mathrm{Re} + iy)\, dy.$$

Summing these four integrals gives the equation:

$$\left(\int_{z.\mathrm{Re}}^{w.\mathrm{Re}} f(x + i\, z.\mathrm{Im})\, dx\right) + i\left(\int_{z.\mathrm{Im}}^{w.\mathrm{Im}} f(w.\mathrm{Re} + iy)\, dy\right) - \left(\int_{z.\mathrm{Re}}^{w.\mathrm{Re}} f(x + i\, w.\mathrm{Im})\, dx\right) - i\left(\int_{z.\mathrm{Im}}^{w.\mathrm{Im}} f(z.\mathrm{Re} + iy)\, dy\right) =$$

Rearranging the terms to match the statement of the lemma concludes the proof. □

Definition 200 (Integral along the taxicab path)**.** Let $0 < R < R_0 < 1$, and assume $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ analyticOnNhd $\overline{\mathbb{D}}_{R_0}$. Define the function $I_f : \overline{\mathbb{D}}_R \to \mathbb{C}$ by

$$I_f(z) := \int_0^{z.\mathrm{Re}} f(t)\, dt + i \int_0^{z.\mathrm{Im}} f(z.\mathrm{Re} + i\tau)\, d\tau.$$

Lemma 201 (Integral form)**.** *Let $0 < R < R_0 < 1$, and assume $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ analyticOnNhd $\overline{\mathbb{D}}_{R_0}$. Let $z \in \overline{\mathbb{D}}_R$ and $h \in \mathbb{C}$ satisfy $z + h \in \overline{\mathbb{D}}_R$. Then*

$$I_f(z + h) = \int_0^{(z+h).\mathrm{Re}} f(t)\, dt + i \int_0^{(z+h).\mathrm{Im}} f((z + h).\mathrm{Re} + i\tau)\, d\tau.$$

Proof. Apply theorem 200 with $z + h$. □

Lemma 202 (Integral form)**.** *Let $0 < R < R_0 < 1$, and assume $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ analyticOnNhd $\overline{\mathbb{D}}_{R_0}$. Let $z \in \overline{\mathbb{D}}_R$. Then*

$$I_f(z) = \int_0^{z.\mathrm{Re}} f(t)\, dt + i \int_0^{z.\mathrm{Im}} f(z.\mathrm{Re} + i\tau)\, d\tau.$$

Proof. Apply theorem 200 with z □

Lemma 203 (Integral form)**.** *Let* $0 < R < R_0 < 1$*, and assume* $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$*. Let* $z \in \overline{\mathbb{D}}_R$*,* $h \in \mathbb{C}$ *satisfy* $z + h \in \overline{\mathbb{D}}_R$*, and let* $w = (z + h).\mathrm{Re} + i\, z.\mathrm{Im}$*. Then*

$$I_f(w) = \int_0^{(z+h).\mathrm{Re}} f(t)\, dt + i \int_0^{z.\mathrm{Im}} f((z + h).\mathrm{Re} + i\tau)\, d\tau.$$

Proof. Apply theorem 200 with w, noting that $w.\mathrm{Re} = (z + h).\mathrm{Re}$ and $w.\mathrm{Im} = z.\mathrm{Im}$. □

Lemma 204 (Difference form)**.** *Let* $0 < R < R_0 < 1$*, and assume* $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$*. Let* $z \in \overline{\mathbb{D}}_R$*,* $h \in \mathbb{C}$ *satisfy* $z + h \in \overline{\mathbb{D}}_R$*, and let* $w = (z + h).\mathrm{Re} + i\, z.\mathrm{Im}$*. Then*

$$I_f(z + h) - I_f(w) = i \int_{z.\mathrm{Im}}^{(z+h).\mathrm{Im}} f((z + h).\mathrm{Re} + i\tau)\, d\tau.$$

Proof. Take the difference of theorem 201 and theorem 203, noting the terms involving $\int f(t)\, dt$ cancel. The remaining terms are combined using properties of integrals. □

Lemma 205 (Initial form)**.** *Let* $0 < R < R_0 < 1$*, and assume* $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$*. Let* $z \in \overline{\mathbb{D}}_R$*,* $h \in \mathbb{C}$ *satisfy* $z + h \in \overline{\mathbb{D}}_R$*, and let* $w = (z + h).\mathrm{Re} + i\, z.\mathrm{Im}$*. Then*

$$I_f(w) - I_f(z) = \int_{z.\mathrm{Re}}^{w.\mathrm{Re}} f(t)\, dt + i \int_0^{z.\mathrm{Im}} [f(w.\mathrm{Re} + i\tau) - f(z.\mathrm{Re} + i\tau)]\, d\tau.$$

Proof. Apply theorem 203 and theorem 202, note that $w.\mathrm{Im} = z.\mathrm{Im}$, and combine integrals. □

Lemma 206 (Horizontal strip)**.** *Let* $0 < R < R_0 < 1$*, and assume* $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$*. Let* $z \in \overline{\mathbb{D}}_R$*,* $h \in \mathbb{C}$ *satisfy* $z + h \in \overline{\mathbb{D}}_R$*, and let* $w = (z + h).\mathrm{Re} + i\, z.\mathrm{Im}$*. Then*

$$\int_{z.\mathrm{Re}}^{w.\mathrm{Re}} f(t)dt - \int_{z.\mathrm{Re}}^{w.\mathrm{Re}} f(t + i\, z.\mathrm{Im})dt + i \int_0^{z.\mathrm{Im}} f(w.\mathrm{Re} + i\tau)d\tau - i \int_0^{z.\mathrm{Im}} f(z.\mathrm{Re} + i\tau)d\tau = 0.$$

Proof. Apply theorem 199 with the points $z' := z.\mathrm{Re}$ and $w' := (z + h).\mathrm{Re} + i\, z.\mathrm{Im}$. The four corners of the rectangle are $z.\mathrm{Re}$, $(z+h).\mathrm{Re}$, $(z+h).\mathrm{Re} + i\, z.\mathrm{Im}$, and $z.\mathrm{Re} + i\, z.\mathrm{Im}$. Substituting z' and w' into the formula from theorem 199 yields the desired identity. □

Lemma 207 (Rearrangement step)**.** *Let* $0 < R < R_0 < 1$*, and assume* $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$*. Let* $z \in \overline{\mathbb{D}}_R$*,* $h \in \mathbb{C}$ *satisfy* $z + h \in \overline{\mathbb{D}}_R$*, and let* $w = (z + h).\mathrm{Re} + i\, z.\mathrm{Im}$*. Then*

$$i \int_0^{z.\mathrm{Im}} [f(w.\mathrm{Re} + i\tau) - f(z.\mathrm{Re} + i\tau)]\, d\tau = \int_{z.\mathrm{Re}}^{w.\mathrm{Re}} f(t + i\, z.\mathrm{Im})\, dt - \int_{z.\mathrm{Re}}^{w.\mathrm{Re}} f(t)\, dt.$$

Proof. We start with the identity from theorem 206. The assumptions are: $0 < R < R_0 < 1$, $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ is analytic on a neighborhood of $\overline{\mathbb{D}}_{R_0}$, $z \in \overline{\mathbb{D}}_R$, $h \in \mathbb{C}$ with $z + h \in \overline{\mathbb{D}}_R$, and $w = (z + h).\mathrm{Re} + i\, z.\mathrm{Im}$. The identity is:

$$\int_{z.\mathrm{Re}}^{w.\mathrm{Re}} f(t)dt - \int_{z.\mathrm{Re}}^{w.\mathrm{Re}} f(t + i\, z.\mathrm{Im})dt + i \int_0^{z.\mathrm{Im}} f(w.\mathrm{Re} + i\tau)d\tau - i \int_0^{z.\mathrm{Im}} f(z.\mathrm{Re} + i\tau)d\tau = 0.$$

By the linearity of integration, we can combine the last two terms:

$$i\int_0^{z.\operatorname{Im}} f(w.\operatorname{Re}+i\tau)d\tau - i\int_0^{z.\operatorname{Im}} f(z.\operatorname{Re}+i\tau)d\tau = i\left(\int_0^{z.\operatorname{Im}} f(w.\operatorname{Re}+i\tau)d\tau - \int_0^{z.\operatorname{Im}} f(z.\operatorname{Re}+i\tau)d\tau\right) = i\int_0^{z.\operatorname{Im}} [f(w$$

Substituting this back into the identity gives:

$$\int_{z.\operatorname{Re}}^{w.\operatorname{Re}} f(t)dt - \int_{z.\operatorname{Re}}^{w.\operatorname{Re}} f(t+i\,z.\operatorname{Im})dt + i\int_0^{z.\operatorname{Im}} [f(w.\operatorname{Re}+i\tau) - f(z.\operatorname{Re}+i\tau)]\,d\tau = 0.$$

To obtain the desired result, we isolate the term involving the integral over τ by moving the other two integral terms to the right-hand side of the equation:

$$i\int_0^{z.\operatorname{Im}} [f(w.\operatorname{Re}+i\tau) - f(z.\operatorname{Re}+i\tau)]\,d\tau = \int_{z.\operatorname{Re}}^{w.\operatorname{Re}} f(t+i\,z.\operatorname{Im})\,dt - \int_{z.\operatorname{Re}}^{w.\operatorname{Re}} f(t)\,dt.$$

This completes the proof. ☐

Lemma 208 (Shift integral)**.** *Let $0 < R < R_0 < 1$, and assume $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ analyticOnNhd $\overline{\mathbb{D}}_{R_0}$. Let $z \in \overline{\mathbb{D}}_R$, $h \in \mathbb{C}$ satisfy $z + h \in \overline{\mathbb{D}}_R$, and let $w = (z+h).\operatorname{Re} + i\,z.\operatorname{Im}$. Then*

$$I_f(w) - I_f(z) = \int_{z.\operatorname{Re}}^{(z+h).\operatorname{Re}} f(t+i\,z.\operatorname{Im})\,dt.$$

Proof. The assumptions are: $0 < R < R_0 < 1$, $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ is analytic on a neighborhood of $\overline{\mathbb{D}}_{R_0}$, $z \in \overline{\mathbb{D}}_R$, $h \in \mathbb{C}$ with $z + h \in \overline{\mathbb{D}}_R$, and $w = (z+h).\operatorname{Re} + i\,z.\operatorname{Im}$. From theorem 205, we have the expression:

$$I_f(w) - I_f(z) = \int_{z.\operatorname{Re}}^{w.\operatorname{Re}} f(t)\,dt + i\int_0^{z.\operatorname{Im}} [f(w.\operatorname{Re}+i\tau) - f(z.\operatorname{Re}+i\tau)]\,d\tau.$$

From theorem 207, we have an identity for the second term in the expression above:

$$i\int_0^{z.\operatorname{Im}} [f(w.\operatorname{Re}+i\tau) - f(z.\operatorname{Re}+i\tau)]\,d\tau = \int_{z.\operatorname{Re}}^{w.\operatorname{Re}} f(t+i\,z.\operatorname{Im})\,dt - \int_{z.\operatorname{Re}}^{w.\operatorname{Re}} f(t)\,dt.$$

We substitute this identity into the expression for $I_f(w) - I_f(z)$:

$$I_f(w) - I_f(z) = \int_{z.\operatorname{Re}}^{w.\operatorname{Re}} f(t)\,dt + \left(\int_{z.\operatorname{Re}}^{w.\operatorname{Re}} f(t+i\,z.\operatorname{Im})\,dt - \int_{z.\operatorname{Re}}^{w.\operatorname{Re}} f(t)\,dt\right).$$

The terms $\int_{z.\operatorname{Re}}^{w.\operatorname{Re}} f(t)\,dt$ and $-\int_{z.\operatorname{Re}}^{w.\operatorname{Re}} f(t)\,dt$ cancel each other out.

$$I_f(w) - I_f(z) = \int_{z.\operatorname{Re}}^{w.\operatorname{Re}} f(t+i\,z.\operatorname{Im})\,dt.$$

Finally, we use the definition of w, which states $w.\operatorname{Re} = (z+h).\operatorname{Re}$. Substituting this into the upper limit of the integral gives the final result:

$$I_f(w) - I_f(z) = \int_{z.\operatorname{Re}}^{(z+h).\operatorname{Re}} f(t+i\,z.\operatorname{Im})\,dt.$$

☐

Lemma 209 (L path)**.** *Let* $0 < R < R_0 < 1$*, and assume* $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$*. Let* $z \in \overline{\mathbb{D}}_R$ *and* $h \in \mathbb{C}$ *satisfy* $z + h \in \overline{\mathbb{D}}_R$*. Then*

$$I_f(z+h) - I_f(z) = \int_{z.\,\mathrm{Re}}^{(z+h).\,\mathrm{Re}} f(t + i\, z.\,\mathrm{Im})\, dt + i \int_{z.\,\mathrm{Im}}^{(z+h).\,\mathrm{Im}} f((z+h).\,\mathrm{Re} + i\tau)\, d\tau.$$

Proof. The result follows by summing the identities from theorem 208 and theorem 204, using the identity $I_f(z+h) - I_f(z) = (I_f(w) - I_f(z)) + (I_f(z+h) - I_f(w))$. □

Lemma 210 (Add-sub step)**.** *Let* $0 < R < R_0 < 1$*, and assume* $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$*. Let* $z \in \overline{\mathbb{D}}_R$ *and* $h \in \mathbb{C}$ *satisfy* $z + h \in \overline{\mathbb{D}}_R$*. Then*

$$I_f(z+h) - I_f(z) = \int_{z.\,\mathrm{Re}}^{(z+h).\,\mathrm{Re}} (f(t + i\, z.\,\mathrm{Im}) - f(z) + f(z))\, dt + i \int_{z.\,\mathrm{Im}}^{(z+h).\,\mathrm{Im}} (f((z+h).\,\mathrm{Re} + i\tau) - f(z) + f(z))\, d\tau.$$

Proof. The identity follows by starting with the expression for $I_f(z+h) - I_f(z)$ from theorem 209 and adding and subtracting the term $f(z)$ within each integrand, which is an algebraic identity. □

Lemma 211 (Linearity split)**.** *Let* $0 < R < R_0 < 1$*, and assume* $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$*. Let* $z \in \overline{\mathbb{D}}_R$ *and* $h \in \mathbb{C}$ *satisfy* $z + h \in \overline{\mathbb{D}}_R$*. Then*

$$\begin{aligned} I_f(z+h) - I_f(z) = &\left(\int_{z.\,\mathrm{Re}}^{(z+h).\,\mathrm{Re}} (f(t + i\, z.\,\mathrm{Im}) - f(z))\, dt + \int_{z.\,\mathrm{Re}}^{(z+h).\,\mathrm{Re}} f(z)\, dt \right) \\ &+ i \left(\int_{z.\,\mathrm{Im}}^{(z+h).\,\mathrm{Im}} (f((z+h).\,\mathrm{Re} + i\tau) - f(z))\, d\tau + \int_{z.\,\mathrm{Im}}^{(z+h).\,\mathrm{Im}} f(z)\, d\tau \right). \end{aligned}$$

Proof. We begin with the identity from theorem 210, which holds under the assumptions that $0 < R < R_0 < 1$, f is analytic on a neighborhood of $\overline{\mathbb{D}}_{R_0}$, $z \in \overline{\mathbb{D}}_R$, and $h \in \mathbb{C}$ with $z + h \in \overline{\mathbb{D}}_R$:

$$I_f(z+h) - I_f(z) = \int_{z.\,\mathrm{Re}}^{(z+h).\,\mathrm{Re}} (f(t + i\, z.\,\mathrm{Im}) - f(z) + f(z))\, dt + i \int_{z.\,\mathrm{Im}}^{(z+h).\,\mathrm{Im}} (f((z+h).\,\mathrm{Re} + i\tau) - f(z) + f(z))\, d\tau.$$

We apply the linearity property of the integral, $\int (g+k) = \int g + \int k$, to each of the two integrals on the right-hand side. For the first integral, we group the integrand as $(f(t + i\, z.\,\mathrm{Im}) - f(z)) + f(z)$. Applying linearity yields:

$$\int_{z.\,\mathrm{Re}}^{(z+h).\,\mathrm{Re}} (f(t + i\, z.\,\mathrm{Im}) - f(z) + f(z))\, dt = \int_{z.\,\mathrm{Re}}^{(z+h).\,\mathrm{Re}} (f(t + i\, z.\,\mathrm{Im}) - f(z))\, dt + \int_{z.\,\mathrm{Re}}^{(z+h).\,\mathrm{Re}} f(z)\, dt.$$

For the second integral, we group the integrand as $(f((z+h).\,\mathrm{Re} + i\tau) - f(z)) + f(z)$. Applying linearity yields:

$$\int_{z.\,\mathrm{Im}}^{(z+h).\,\mathrm{Im}} (f((z+h).\,\mathrm{Re} + i\tau) - f(z) + f(z))\, d\tau = \int_{z.\,\mathrm{Im}}^{(z+h).\,\mathrm{Im}} (f((z+h).\,\mathrm{Re} + i\tau) - f(z))\, d\tau + \int_{z.\,\mathrm{Im}}^{(z+h).\,\mathrm{Im}} f(z)\, d\tau.$$

Substituting these expanded forms back into the original equation for $I_f(z+h) - I_f(z)$, and distributing the factor of i for the second part, we obtain the desired result:

$$\begin{aligned} I_f(z+h) - I_f(z) = &\left(\int_{z.\operatorname{Re}}^{(z+h).\operatorname{Re}} (f(t + i\, z.\operatorname{Im}) - f(z))\, dt + \int_{z.\operatorname{Re}}^{(z+h).\operatorname{Re}} f(z)\, dt \right) \\ &+ i \left(\int_{z.\operatorname{Im}}^{(z+h).\operatorname{Im}} (f((z+h).\operatorname{Re} + i\tau) - f(z))\, d\tau + \int_{z.\operatorname{Im}}^{(z+h).\operatorname{Im}} f(z)\, d\tau \right). \end{aligned}$$

□

Lemma 212 (Integral of constant over L-path)**.** *Let $0 < R < R_0 < 1$, and assume $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ analyticOnNhd $\overline{\mathbb{D}}_{R_0}$. Let $z \in \overline{\mathbb{D}}_R$ and $h \in \mathbb{C}$ satisfy $z + h \in \overline{\mathbb{D}}_R$. Then*

$$\int_{z.\operatorname{Re}}^{(z+h).\operatorname{Re}} f(z)\, dt + i \int_{z.\operatorname{Im}}^{(z+h).\operatorname{Im}} f(z)\, d\tau = f(z) \cdot h.$$

Proof. The left side is the integral of the constant function $w \mapsto f(z)$ over the L-shaped path. Thus we calculate

$$\begin{aligned} \int_{z.\operatorname{Re}}^{(z+h).\operatorname{Re}} f(z)\, dt + i \int_{z.\operatorname{Im}}^{(z+h).\operatorname{Im}} f(z)\, d\tau &= f(z) \cdot ((z+h).\operatorname{Re} - z.\operatorname{Re}) + i \cdot f(z) \cdot ((z+h).\operatorname{Im} - z.\operatorname{Im}) \\ &= f(z) \cdot (h.\operatorname{Re} + i \cdot h.\operatorname{Im}) = f(z) \cdot h. \end{aligned}$$

□

Lemma 213 (Difference decomposition)**.** *Let $0 < R < R_0 < 1$, and assume $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ analyticOnNhd $\overline{\mathbb{D}}_{R_0}$. Let $z \in \overline{\mathbb{D}}_R$ and $h \in \mathbb{C}$ satisfy $z + h \in \overline{\mathbb{D}}_R$. Then*

$$I_f(z+h) - I_f(z) = h \cdot f(z) + \operatorname{Err}(z, h),$$

where $\operatorname{Err}(z, h)$ *is defined as*

$$\operatorname{Err}(z, h) := \int_{z.\operatorname{Re}}^{(z+h).\operatorname{Re}} (f(t + i\, z.\operatorname{Im}) - f(z))\, dt + i \int_{z.\operatorname{Im}}^{(z+h).\operatorname{Im}} (f((z+h).\operatorname{Re} + i\tau) - f(z))\, d\tau.$$

Proof. We start with the expression for $I_f(z+h) - I_f(z)$ from theorem 211. The assumptions are that f is analytic on a neighborhood of $\overline{\mathbb{D}}_{R_0}$, $z \in \overline{\mathbb{D}}_R$, and $h \in \mathbb{C}$ such that $z + h \in \overline{\mathbb{D}}_R$.

$$I_f(z+h) - I_f(z) = \left(\int_{z.\operatorname{Re}}^{(z+h).\operatorname{Re}} (f(t + i\, z.\operatorname{Im}) - f(z))\, dt + \int_{z.\operatorname{Re}}^{(z+h).\operatorname{Re}} f(z)\, dt \right) + i \left(\int_{z.\operatorname{Im}}^{(z+h).\operatorname{Im}} (f((z+h).\operatorname{Re} + i\tau) - f(z)$$

We can rearrange the terms by grouping them differently:

$$\begin{aligned} I_f(z+h) - I_f(z) = &\left(\int_{z.\operatorname{Re}}^{(z+h).\operatorname{Re}} (f(t + i\, z.\operatorname{Im}) - f(z))\, dt + i \int_{z.\operatorname{Im}}^{(z+h).\operatorname{Im}} (f((z+h).\operatorname{Re} + i\tau) - f(z))\, d\tau \right) \\ &+ \left(\int_{z.\operatorname{Re}}^{(z+h).\operatorname{Re}} f(z)\, dt + i \int_{z.\operatorname{Im}}^{(z+h).\operatorname{Im}} f(z)\, d\tau \right). \end{aligned}$$

The first large grouped term is precisely the definition of $\mathrm{Err}(z,h)$ given in the lemma statement. The second large grouped term is an expression that is evaluated in theorem 212. According to that lemma,

$$\int_{z.\mathrm{Re}}^{(z+h).\mathrm{Re}} f(z)\,dt + i\int_{z.\mathrm{Im}}^{(z+h).\mathrm{Im}} f(z)\,d\tau = f(z)\cdot h.$$

Substituting these two results back into our rearranged equation, we get:

$$I_f(z+h) - I_f(z) = \mathrm{Err}(z,h) + f(z)\cdot h.$$

Swapping the terms on the right-hand side gives the final statement. □

Lemma 214 (Bound on error term)**.** *Let* $0 < R < R_0 < 1$*, and assume* $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$*. Let* $z \in \overline{\mathbb{D}}_R$ *and* $h \in \mathbb{C}$ *satisfy* $z+h \in \overline{\mathbb{D}}_R$ *and* $z-h \in \overline{\mathbb{D}}_R$*. Let*

$$S_{horiz}(z,h) := \sup_{z.\mathrm{Re}-|h.\mathrm{Re}|\le t\le z.\mathrm{Re}+|h.\mathrm{Re}|} |f(t+i\,z.\mathrm{Im}) - f(z)|$$

$$S_{vert}(z,h) := \sup_{z.\mathrm{Im}-|h.\mathrm{Im}|\le \tau\le z.\mathrm{Im}+|h.\mathrm{Im}|} |f((z+h).\mathrm{Re}+i\tau) - f(z)|.$$

$$S(z,h) := \max(S_{horiz}(z,h), S_{vert}(z,h))$$

Then the error term $\mathrm{Err}(z,h)$ *is bounded by:*

$$|\mathrm{Err}(z,h)| \le |h.\mathrm{Re}|S(z,h) + |h.\mathrm{Im}|S(z,h).$$

Proof. We begin with the definition of $\mathrm{Err}(z,h)$ from theorem 213.

$$\mathrm{Err}(z,h) = \int_{z.\mathrm{Re}}^{(z+h).\mathrm{Re}} (f(t+i\,z.\mathrm{Im}) - f(z))\,dt + i\int_{z.\mathrm{Im}}^{(z+h).\mathrm{Im}} (f((z+h).\mathrm{Re}+i\tau) - f(z))\,d\tau.$$

We take the modulus and apply the triangle inequality, $|A+B| \le |A| + |B|$:

$$|\mathrm{Err}(z,h)| \le \left|\int_{z.\mathrm{Re}}^{(z+h).\mathrm{Re}} (f(t+i\,z.\mathrm{Im}) - f(z))\,dt\right| + \left|i\int_{z.\mathrm{Im}}^{(z+h).\mathrm{Im}} (f((z+h).\mathrm{Re}+i\tau) - f(z))\,d\tau\right|.$$

Since $|i| = 1$, the second term simplifies to $\left|\int_{z.\mathrm{Im}}^{(z+h).\mathrm{Im}} (f((z+h).\mathrm{Re}+i\tau) - f(z))\,d\tau\right|$. Now we apply the ML-inequality ($|\int_\gamma g(\zeta)d\zeta| \le \mathrm{length}(\gamma)\cdot\sup_{\zeta\in\gamma}|g(\zeta)|$) to each integral.

For the first integral, the path of integration is the line segment from $z.\mathrm{Re}$ to $(z+h).\mathrm{Re}$. The length of this path is $|(z+h).\mathrm{Re} - z.\mathrm{Re}| = |h.\mathrm{Re}|$. The supremum of the integrand's modulus is taken over this path. The integration variable t is in the interval between $z.\mathrm{Re}$ and $z.\mathrm{Re}+h.\mathrm{Re}$. This interval is contained within $[z.\mathrm{Re}-|h.\mathrm{Re}|, z.\mathrm{Re}+|h.\mathrm{Re}|]$. Therefore, the supremum over the integration path is less than or equal to the supremum over this larger interval, which is $S_{horiz}(z,h)$.

$$\left|\int_{z.\mathrm{Re}}^{(z+h).\mathrm{Re}} (f(t+i\,z.\mathrm{Im}) - f(z))\,dt\right| \le |h.\mathrm{Re}|\cdot \sup_{t \text{ between } z.\mathrm{Re},(z+h).\mathrm{Re}} |f(t+i\,z.\mathrm{Im}) - f(z)| \le |h.\mathrm{Re}|\cdot S_{horiz}(z,h).$$

For the second integral, the path is from $z.\mathrm{Im}$ to $(z+h).\mathrm{Im}$, with length $|(z+h).\mathrm{Im} - z.\mathrm{Im}| = |h.\mathrm{Im}|$. Similarly, the supremum of its integrand's modulus is bounded by $S_{vert}(z,h)$.

$$\left|\int_{z.\mathrm{Im}}^{(z+h).\mathrm{Im}} (f((z+h).\mathrm{Re}+i\tau) - f(z))\,d\tau\right| \le |h.\mathrm{Im}|\cdot S_{vert}(z,h).$$

Combining these inequalities, we get:

$$|\mathrm{Err}(z,h)| \le |h.\mathrm{Re}\,|S_{horiz}(z,h) + |h.\mathrm{Im}\,|S_{vert}(z,h).$$

By definition, $S(z,h) = \max(S_{horiz}(z,h), S_{vert}(z,h))$. Thus, $S_{horiz}(z,h) \le S(z,h)$ and $S_{vert}(z,h) \le S(z,h)$. Substituting these into the inequality gives:

$$|\mathrm{Err}(z,h)| \le |h.\mathrm{Re}\,|S(z,h) + |h.\mathrm{Im}\,|S(z,h).$$

This is the desired result. □

Lemma 215 (Bound on error term ratio)**.** *Let $0 < R < R_0 < 1$, and assume $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ analyticOnNhd $\overline{\mathbb{D}}_{R_0}$. Let $z \in \overline{\mathbb{D}}_R$ and $h \in \mathbb{C}$ satisfy $z + h \in \overline{\mathbb{D}}_R$ and $z - h \in \overline{\mathbb{D}}_R$. Let $S(z,h)$ be defined as in theorem 214. If $h \neq 0$ then*

$$\left|\frac{\mathrm{Err}(z,h)}{h}\right| \le 2S(z,h).$$

Proof. We start with the inequality from theorem 214, which holds under the given assumptions.

$$|\mathrm{Err}(z,h)| \le |h.\mathrm{Re}\,|S(z,h) + |h.\mathrm{Im}\,|S(z,h) = (|h.\mathrm{Re}\,| + |h.\mathrm{Im}\,|)S(z,h).$$

The lemma includes the explicit assumption that $h \neq 0$, which implies $|h| > 0$. We can therefore divide the inequality by $|h|$ without changing the direction of the inequality.

$$\frac{|\mathrm{Err}(z,h)|}{|h|} \le \frac{|h.\mathrm{Re}\,| + |h.\mathrm{Im}\,|}{|h|} S(z,h).$$

Using the property that $|\frac{A}{B}| = \frac{|A|}{|B|}$ for complex numbers, the left side is equal to $\left|\frac{\mathrm{Err}(z,h)}{h}\right|$. For any complex number $h = h.\mathrm{Re} + ih.\mathrm{Im}$, we know that $|h.\mathrm{Re}\,| \le \sqrt{(h.\mathrm{Re})^2 + (h.\mathrm{Im})^2} = |h|$ and $|h.\mathrm{Im}\,| \le \sqrt{(h.\mathrm{Re})^2 + (h.\mathrm{Im})^2} = |h|$. Therefore, the sum is bounded: $|h.\mathrm{Re}\,| + |h.\mathrm{Im}\,| \le |h| + |h| = 2|h|$. This gives us a bound for the fraction:

$$\frac{|h.\mathrm{Re}\,| + |h.\mathrm{Im}\,|}{|h|} \le \frac{2|h|}{|h|} = 2.$$

Substituting this bound back into our main inequality, we get:

$$\left|\frac{\mathrm{Err}(z,h)}{h}\right| \le 2S(z,h).$$

This completes the proof. □

Lemma 216 (Limit of $S(z,h)$ is zero)**.** *Let $0 < R < R_0 < 1$, and assume $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ analyticOnNhd $\overline{\mathbb{D}}_{R_0}$. Let $z \in \overline{\mathbb{D}}_R$. Then with $S(z,h)$ defined as in theorem 214, we have*

$$\lim_{h \to 0} S(z,h) = 0.$$

Proof. The assumption that f is analytic on a neighborhood of $\overline{\mathbb{D}}_{R_0}$ implies that f is continuous at every point in $\overline{\mathbb{D}}_{R_0}$. In particular, f is continuous at $z \in \overline{\mathbb{D}}_R$. By the definition of continuity at z, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any point w satisfying $|w - z| < \delta$, we have $|f(w) - f(z)| < \epsilon$.

We want to show that $\lim_{h\to 0} S(z,h) = 0$. By definition, $S(z,h) = \max(S_{horiz}(z,h), S_{vert}(z,h))$. The limit will be zero if we can show that both $S_{horiz}(z,h)$ and $S_{vert}(z,h)$ tend to zero.

1. **Analysis of** $S_{horiz}(z,h)$: $S_{horiz}(z,h) = \sup_{t\in[z.\mathrm{Re}-|h.\mathrm{Re}|, z.\mathrm{Re}+|h.\mathrm{Re}|]} |f(t+i\,z.\mathrm{Im}) - f(z)|$. Let $w_t = t + i\,z.\mathrm{Im}$ be a point on the horizontal segment. We need to bound $|w_t - z|$. $|w_t - z| = |(t+i\,z.\mathrm{Im})-(z.\mathrm{Re}+i\,z.\mathrm{Im})| = |t-z.\mathrm{Re}|$. The supremum is over t such that $|t-z.\mathrm{Re}| \le |h.\mathrm{Re}|$. Since $|h.\mathrm{Re}| \le |h|$, we have $|w_t - z| \le |h|$. If we choose $|h| < \delta$, then for all t in the interval, $|w_t - z| < \delta$. By the continuity of f, this implies $|f(w_t)-f(z)| < \epsilon$. Since this is true for all values in the set, their supremum must be less than or equal to ϵ. Thus, for $|h| < \delta$, $S_{horiz}(z,h) \le \epsilon$.

2. **Analysis of** $S_{vert}(z,h)$: $S_{vert}(z,h) = \sup_{\tau\in[z.\mathrm{Im}-|h.\mathrm{Im}|, z.\mathrm{Im}+|h.\mathrm{Im}|]} |f((z+h).\mathrm{Re}+i\tau) - f(z)|$. Let $w_\tau = (z+h).\mathrm{Re}+i\tau = (z.\mathrm{Re}+h.\mathrm{Re}) + i\tau$ be a point on the vertical segment. We bound $|w_\tau - z|$. $|w_\tau - z| = |(z.\mathrm{Re}+h.\mathrm{Re}+i\tau)-(z.\mathrm{Re}+iz.\mathrm{Im})| = |h.\mathrm{Re}+i(\tau-z.\mathrm{Im})|$. Using the triangle inequality, $|w_\tau - z| \le |h.\mathrm{Re}|+|i(\tau-z.\mathrm{Im})| = |h.\mathrm{Re}|+|\tau-z.\mathrm{Im}|$. The supremum is over τ such that $|\tau-z.\mathrm{Im}| \le |h.\mathrm{Im}|$. So, $|w_\tau - z| \le |h.\mathrm{Re}|+|h.\mathrm{Im}|$. We know $|h.\mathrm{Re}|+|h.\mathrm{Im}| \le 2|h|$. If we choose $|h| < \delta/2$, then $|w_\tau - z| \le 2|h| < \delta$. By continuity, $|f(w_\tau) - f(z)| < \epsilon$. Thus, for $|h| < \delta/2$, $S_{vert}(z,h) \le \epsilon$.

Given $\epsilon > 0$, we can choose $\delta' = \delta/2$. Then for any h with $|h| < \delta'$, both $S_{horiz}(z,h) \le \epsilon$ and $S_{vert}(z,h) \le \epsilon$. Therefore, $S(z,h) = \max(S_{horiz}(z,h), S_{vert}(z,h)) \le \epsilon$ for all $|h| < \delta'$. This satisfies the definition of the limit, so $\lim_{h\to 0} S(z,h) = 0$. □

Lemma 217 (Limit of error term ratio is zero)**.** *Let* $0 < R < R_0 < 1$*, and assume* $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$*. Let* $z \in \overline{\mathbb{D}}_R$*. Then*

$$\lim_{h\to 0} \frac{\mathrm{Err}(z,h)}{h} = 0.$$

Proof. To prove the limit, we will use the Squeeze Theorem. The limit is taken as $h \to 0$, so we consider $h \neq 0$. From theorem 215, we have the inequality for the modulus of the error term ratio:

$$\left|\frac{\mathrm{Err}(z,h)}{h}\right| \le 2S(z,h).$$

The modulus of any complex number is non-negative, so we can write:

$$0 \le \left|\frac{\mathrm{Err}(z,h)}{h}\right| \le 2S(z,h).$$

Now, we take the limit of all parts of the inequality as $h \to 0$. The lower bound is constant, so $\lim_{h\to 0} 0 = 0$. For the upper bound, we use theorem 216, which states that $\lim_{h\to 0} S(z,h) = 0$. Therefore, $\lim_{h\to 0} 2S(z,h) = 2 \cdot (\lim_{h\to 0} S(z,h)) = 2 \cdot 0 = 0$. Since $\left|\frac{\mathrm{Err}(z,h)}{h}\right|$ is squeezed between two functions that both tend to 0 as $h \to 0$, the Squeeze Theorem (Mathlib: Filter.Tendsto.squeeze') implies that the limit of the modulus is also 0:

$$\lim_{h\to 0} \left|\frac{\mathrm{Err}(z,h)}{h}\right| = 0.$$

A sequence of complex numbers converges to 0 if and only if the sequence of their moduli converges to 0. Therefore, we can conclude that:

$$\lim_{h\to 0} \frac{\mathrm{Err}(z,h)}{h} = 0.$$

□

Lemma 218 (Differentiability of $I_f(z)$)**.** *Let $0 < R < R_0 < 1$, and assume $f : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ analyticOnNhd $\overline{\mathbb{D}}_{R_0}$. The function $I_f(z)$ is analyticOnNhd $\overline{\mathbb{D}}_R$, and $I_f'(z) = f(z)$ on $\overline{\mathbb{D}}_R$.*

Proof. To show that $I_f(z)$ is differentiable at a point $z \in \overline{\mathbb{D}}_R$ and that its derivative is $f(z)$, we must show that the following limit exists and equals $f(z)$:

$$I_f'(z) = \lim_{h \to 0} \frac{I_f(z+h) - I_f(z)}{h}.$$

We use the decomposition from theorem 213, which states:

$$I_f(z+h) - I_f(z) = h \cdot f(z) + \mathrm{Err}(z,h).$$

For $h \neq 0$, we can form the difference quotient by dividing by h:

$$\frac{I_f(z+h) - I_f(z)}{h} = \frac{h \cdot f(z) + \mathrm{Err}(z,h)}{h} = f(z) + \frac{\mathrm{Err}(z,h)}{h}.$$

Now, we take the limit as $h \to 0$:

$$I_f'(z) = \lim_{h \to 0} \left(f(z) + \frac{\mathrm{Err}(z,h)}{h} \right).$$

Using the property that the limit of a sum is the sum of the limits:

$$I_f'(z) = \lim_{h \to 0} f(z) + \lim_{h \to 0} \frac{\mathrm{Err}(z,h)}{h}.$$

The term $f(z)$ is constant with respect to h, so its limit is $f(z)$. From theorem 217, we know that $\lim_{h \to 0} \frac{\mathrm{Err}(z,h)}{h} = 0$. Substituting these results back, we find:

$$I_f'(z) = f(z) + 0 = f(z).$$

This shows that for any $z \in \overline{\mathbb{D}}_R$, the derivative $I_f'(z)$ exists and is equal to $f(z)$. Since f is analytic on a neighborhood of $\overline{\mathbb{D}}_{R_0}$, it is continuous on that neighborhood. This means $I_f'(z) = f(z)$ is continuous on $\overline{\mathbb{D}}_R$. A function with a continuous derivative is analytic. To show it is 'analyticOnNhd' $\overline{\mathbb{D}}_R$, we note that since $R < R_0$, we can choose an R' such that $R < R' < R_0$. The entire construction and proof holds for any $z \in \overline{\mathbb{D}}_{R'}$. This shows that I_f is differentiable in the open disk $\mathbb{D}_{R'}$, which is an open neighborhood of $\overline{\mathbb{D}}_R$. Therefore, I_f is analytic on a neighborhood of $\overline{\mathbb{D}}_R$. ☐

1.4 Complex logarithm

Lemma 219 (Logarithmic derivative is analytic)**.** *Let $0 < R < R_0 < 1$, and assume $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ is analyticOnNhd $\overline{\mathbb{D}}_{R_0}$ and $B(z) \neq 0$ for all $z \in \overline{\mathbb{D}}_{R_0}$. Then the function $B'(z)/B(z)$ is analyticOnNhd $\overline{\mathbb{D}}_{R_0}$.*

Proof. Mathlib: AnalyticOnNhd.div ☐

Lemma 220 (Antiderivative of logarithmic derivative)**.** *Let $0 < R < R_0 < 1$, and assume $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ is analyticOnNhd $\overline{\mathbb{D}}_{R_0}$ and $B(z) \neq 0$ for all $z \in \overline{\mathbb{D}}_{R_0}$. Then there exists $J : \overline{\mathbb{D}}_R \to \mathbb{C}$ analyticOnNhd $\overline{\mathbb{D}}_R$, such that $J(0) = 0$ and $J'(z) = B'(z)/B(z)$ for all $z \in \overline{\mathbb{D}}_R$.*

Proof. Take $J = I_{B'/B}$ from theorem 218. Here B/B' is analyticOnNhd $\overline{\mathbb{D}}_{R_0}$ by theorem 219. $\square$

Definition 221 (Auxiliary function)**.** Let $0 < R < R_0 < 1$, and assume $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ is analyticOnNhd $\overline{\mathbb{D}}_{R_0}$ and $B(z) \neq 0$ for all $z \in \overline{\mathbb{D}}_{R_0}$. Define $J(z) := I_{B'/B}(z)$ from theorem 220. Define $H(z) := \exp(J(z))/B(z)$.

Lemma 222 (Exponential of I_f at zero)**.** *Let $0 < R < R_0 < 1$, and assume $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ is analyticOnNhd $\overline{\mathbb{D}}_{R_0}$ and $B(z) \neq 0$ for all $z \in \overline{\mathbb{D}}_{R_0}$. Let J be from theorem 220. Then* $\exp(J(0)) = 1$.

Proof. By theorem 220, we have $J(0) = 0$. Then $e^0 = 1$ by Mathlib: Complex.exp_zero. $\square$

Lemma 223 (Value of H at zero)**.** *Let $0 < R < R_0 < 1$, assume $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ is analyticOnNhd $\overline{\mathbb{D}}_{R_0}$ and $B(z) \neq 0$ for all $z \in \overline{\mathbb{D}}_{R_0}$. Let H be the function from theorem 221. Then $H(0) = 1/B(0)$.*

Proof. By theorem 221 at $z = 0$, $H(0) = \exp(J(0))/B(0)$. Then apply theorem 222. $\square$

Lemma 224 (Logarithmic derivative identity)**.** *Let $0 < R < R_0 < 1$, and assume $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ is analyticOnNhd $\overline{\mathbb{D}}_{R_0}$ and $B(z) \neq 0$ for all $z \in \overline{\mathbb{D}}_{R_0}$. For J from theorem 220, then $J'(z)B(z) = B'(z)$ for all $z \in \overline{\mathbb{D}}_R$.*

Proof. Apply theorem 220. Since $B(z) \neq 0$, multiply by $B(z)$. $\square$

Lemma 225 (Logarithmic derivative identity)**.** *Let $0 < R < R_0 < 1$, and assume $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ is analyticOnNhd $\overline{\mathbb{D}}_{R_0}$ and $B(z) \neq 0$ for all $z \in \overline{\mathbb{D}}_{R_0}$. For J from theorem 220, then $J'(z)B(z) - B'(z) = 0$ for all $z \in \overline{\mathbb{D}}_R$.*

Proof. By theorem 224. $\square$

Lemma 226 (Derivative of $H(z)$)**.** *Let $0 < R < R_0 < 1$, assume $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ is analyticOnNhd $\overline{\mathbb{D}}_{R_0}$ and $B(z) \neq 0$ for all $z \in \overline{\mathbb{D}}_{R_0}$. Let J and H be the functions from theorem 221. The derivative of $H(z)$ is given by*

$$H'(z) = \frac{(\exp(J(z)))' \cdot B(z) - B'(z) \cdot \exp(J(z))}{B(z)^2}.$$

Proof. Apply Mathlib: deriv_div to $H(z) = \exp(J(z))/B(z)$. $B(z) \neq 0$ by assumption. $\square$

Lemma 227 (Derivative of $\exp(J(z))$)**.** *Let $0 < R < R_0 < 1$, and assume $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ is analyticOnNhd $\overline{\mathbb{D}}_{R_0}$ and $B(z) \neq 0$ for all $z \in \overline{\mathbb{D}}_{R_0}$. For J from theorem 220, then*

$$(\exp(J(z)))' = J'(z) \cdot \exp(J(z)).$$

Proof. Apply Mathlib: deriv.scomp_of_eq and AnalyticAt.differentiableAt to the composition $\exp \circ I$. Here J analyticOnNhd $\overline{\mathbb{D}}_R$ by theorem 220. $\square$

Lemma 228 (Derivative of $H(z)$)**.** *Let $0 < R < R_0 < 1$, assume $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ is analyticOnNhd $\overline{\mathbb{D}}_{R_0}$ and $B(z) \neq 0$ for all $z \in \overline{\mathbb{D}}_{R_0}$. Let J and H be the functions from theorem 221. The derivative of $H(z)$ is given by*

$$H'(z) = \frac{(J'(z)B(z) - B'(z)) \exp(J(z))}{B(z)^2}.$$

Proof. By theorems 226 and 227. ☐

Lemma 229 (Derivative of $H(z)$ is 0)**.** *Let* $0 < R < R_0 < 1$*, assume* $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *is analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$ *and* $B(z) \neq 0$ *for all* $z \in \overline{\mathbb{D}}_{R_0}$*. Let* J *and* H *be the functions from theorem 221. Then* $H'(z) = 0$ *for all* $z \in \overline{\mathbb{D}}_R$*.*

Proof. Apply theorems 225 and 228. ☐

Lemma 230 ($H(z)$ is constant)**.** *Let* $0 < R < R_0 < 1$*, assume* $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *is analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$ *and* $B(z) \neq 0$ *for all* $z \in \overline{\mathbb{D}}_{R_0}$*. Let* H *be from theorem 221. Then* $H(z) = H(0)$ *for all* $z \in \overline{\mathbb{D}}_R$*.*

Proof. Apply Mathlib: is_const_of_fderiv_eq_zero to theorem 229 with $H(z)$ on the connected set $\overline{\mathbb{D}}_R$. ☐

Lemma 231 ($H = 1$)**.** *Let* $0 < R < R_0 < 1$*, assume* $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *is analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$*,* $B(z) \neq 0$ *for all* $z \in \overline{\mathbb{D}}_{R_0}$*. Let* H *be the function from theorem 221. Then* $H(z) = 1/B(0)$ *for all* $z \in \overline{\mathbb{D}}_R$*.*

Proof. Apply theorems 223 and 230. ☐

Lemma 232 (Existence of analytic logarithm)**.** *Let* $0 < R < R_0 < 1$*, and assume* $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *is analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$ *with* $B(z) \neq 0$ *for all* $z \in \overline{\mathbb{D}}_{R_0}$*. For* J *from theorem 220, then* $B(z) = B(0)\exp(J(z))$ *for all* $z \in \overline{\mathbb{D}}_R$*.*

Proof. By theorems 221 and 231. ☐

Lemma 233 (Modulus of $\exp(J(z))$)**.** *Let* $0 < R < R_0 < 1$*, and assume* $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *is analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$ *and* $B(z) \neq 0$ *for all* $z \in \overline{\mathbb{D}}_{R_0}$*. Let* J *be the function from theorem 220. Then for any* $z \in \overline{\mathbb{D}}_R$*,*

$$|\exp(J(z))| = \exp(\Re(J(z))).$$

Proof. Apply Mathlib: Complex.abs_exp with $w = J(z)$. ☐

Lemma 234 (Modulus of $B(z)$ in product form)**.** *Let* $0 < R < R_0 < 1$*, assume* $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *is analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$ *with* $B(z) \neq 0$ *for all* $z \in \overline{\mathbb{D}}_{R_0}$*. Let* J *be the function from theorem 220. Then* $|B(z)| = |B(0)| \cdot |e^{J(z)}|$*.*

Proof. By theorem 232, then apply modulus and Mathlib: abs_mul. ☐

Lemma 235 (Modulus of $\exp(J(z))$)**.** *Let* $0 < R < R_0 < 1$*, assume* $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *is analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$ *with* $B(z) \neq 0$ *for all* $z \in \overline{\mathbb{D}}_{R_0}$*. Let* J *be the function from theorem 220. Then* $|B(z)| = |B(0)| \cdot e^{\Re(J(z))}$*.*

Proof. By theorems 233 and 234. ☐

Lemma 236 (Logarithm of modulus as sum)**.** *Let* $0 < R < R_0 < 1$*, assume* $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *is analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$ *with* $B(z) \neq 0$ *for all* $z \in \overline{\mathbb{D}}_{R_0}$*. Let* J *be the function from theorem 220. Then* $\log|B(z)| = \log|B(0)| + \log(e^{\Re(J(z))})$*.*

Proof. Apply theorem 235 then Mathlib: Real.log_mul. ☐

Lemma 237 (Real logarithm of modulus difference)**.** *Let* $0 < R < R_0 < 1$*, assume* $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *is analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$ *with* $B(z) \neq 0$ *for all* $z \in \overline{\mathbb{D}}_{R_0}$*. Let* J *be the function from theorem 220. Then* $\log|B(z)| - \log|B(0)| = \Re(J(z))$*.*

Proof. Apply theorem 236 and Mathlib: Real.log_exp to $x = \Re(J(z))$. □

Lemma 238 (Logarithm of an analytic function)**.** *Let* $0 < R < R_0 < 1$*, assume* $B : \overline{\mathbb{D}}_{R_0} \to \mathbb{C}$ *is analyticOnNhd* $\overline{\mathbb{D}}_{R_0}$ *with* $B(z) \neq 0$ *for all* $z \in \overline{\mathbb{D}}_{R_0}$*. Then there exists* $J_B : \overline{\mathbb{D}}_R \to \mathbb{C}$ *analyticOnNhd* $\overline{\mathbb{D}}_R$*, such that* $J_B(0) = 0$*, and* $J_B'(z) = B'(z)/B(z)$ *and* $\log|B(z)| - \log|B(0)| = \Re(J_B(z))$ *for all* $z \in \overline{\mathbb{D}}_R$*.*

Proof. Apply theorems 220 and 237. □

Chapter 2

Log Derivative

Lemma 239 (Disk inclusion). *Let* $0 < R < 1$. *Then we have* $\overline{\mathbb{D}}_R \subset \mathbb{D}_1$.

Proof. Unfold definitions of $\overline{\mathbb{D}}_R$ and $\mathbb{D}_1$. Calculate $|z| \leq R < 1$. □

Definition 240 (Zero set). Let $R > 0$ and $f : \overline{\mathbb{D}}_R \to \mathbb{C}$. Define the set of zeros $\mathcal{K}_f(R) = \{\rho \in \mathbb{C} : |\rho| \leq R \text{ and } f(\rho) = 0\}$.

Lemma 241 (Zero containment). *Let* $R > 0$ *and* $f : \overline{\mathbb{D}}_R \to \mathbb{C}$. *Then we have* $\mathcal{K}_f(R) \subset \overline{\mathbb{D}}_R$.

Proof. Unfold definition of $\mathcal{K}_f(R)$. □

Lemma 242 (Zero in disk). *Let* $0 < R < 1$ *and* $f : \mathbb{D}_1 \to \mathbb{C}$. *Then we have* $\mathcal{K}_f(R) \subset \{\rho \in \mathbb{D}_1 : f(\rho) = 0\}$.

Proof. Unfold definition of $\mathcal{K}_f(R)$. □

Lemma 243 (Accumulation point). *Let* $D \subset \mathbb{C}$ *be a compact set. If* $Z \subset D$ *is an infinite subset, then* Z *has an accumulation point* $\rho_0 \in D$.

Proof. □

Lemma 244 (Zeros accumulate). *Let* $R > 0$ *and* $f : \overline{\mathbb{D}}_R \to \mathbb{C}$. *If* $\mathcal{K}_f(R) \subset \overline{\mathbb{D}}_R$ *is infinite, then* $\mathcal{K}_f(R)$ *has an accumulation point* $\rho_0 \in \overline{\mathbb{D}}_R$.

Proof. Apply Lemmas 99, 241, 243 with $D = \overline{\mathbb{D}}_R$ and $Z = \mathcal{K}_f(R)$. □

Lemma 245 (Identity theorem). *Let* $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ *be analytic. Suppose there exists* $\rho_0 \in \mathbb{D}_1$ *an accumulation point of* $\{\rho \in \mathbb{D}_1 : f(\rho) = 0\}$. *Then* $f(z) = 0$ *for all* $z \in \mathbb{D}_1$.

Proof. □

Lemma 246 (Identity theorem R). *Let* $0 < R < 1$ *and* $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ *be analytic. Suppose there exists* $\rho_0 \in \overline{\mathbb{D}}_R$ *an accumulation point of* $\{\rho \in \mathbb{D}_1 : f(\rho) = 0\}$. *Then* $f(z) = 0$ *for all* $z \in \mathbb{D}_1$.

Proof. Apply Lemmas 245 and 239. □

Lemma 247 (Identity on K). *Let* $0 < R < 1$ *and* $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ *be analytic. Suppose there exists* $\rho_0 \in \overline{\mathbb{D}}_R$ *an accumulation point of* $\mathcal{K}_f(R)$. *Then* $f(z) = 0$ *for all* $z \in \mathbb{D}_1$.

Proof. Apply Lemmas 246 and 242. □

Lemma 248 (Infinite zeros imply)**.** *Let $0 < R < 1$ and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be analytic. If $\mathcal{K}_f(R)$ is infinite, then $f(z) = 0$ for all $z \in \mathbb{D}_1$.*

Proof. Apply Lemmas 247 and 244. □

Lemma 249 (Finite zeros)**.** *Let $0 < R < 1$ and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be analytic. If there exists $z \in \mathbb{D}_1$ such that $f(z) \neq 0$, then $\mathcal{K}_f(R)$ is finite.*

Proof. Contrapositive of Lemma 248. □

2.1 B_f analytic and never zero

Definition 250 (Zero order)**.** Let $0 < R_1 < 1$ and $f : \mathbb{C} \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$. For any zero $\rho \in \mathcal{K}_f(R_1)$ of the function f, we define $m_{\rho,f}$ as the analytic order of f at ρ, denoted by analyticOrderAt f ρ.

Lemma 251 (Order is integer)**.** *Let $0 < R < 1$, $R_1 = \frac{2}{3}R$, and $f : \mathbb{C} \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$. If $f(0) \neq 0$ then $m_{\rho,f} \in \mathbb{N}$ for all $\rho \in \mathcal{K}_f(R_1)$.*

Proof. Let ρ be an arbitrary element of $\mathcal{K}_f(R_1)$. By the definition of $\mathcal{K}_f(R_1)$ (see theorem 240), any element $\rho \in \mathcal{K}_f(R_1)$ is a zero of f, which means $f(\rho) = 0$. The function f is assumed to be 'AnalyticOnNhd' on $\overline{\mathbb{D}}_1$. This implies that for any point $w \in \overline{\mathbb{D}}_1$, there exists an open neighborhood of w where f is analytic. Since $\rho \in \mathcal{K}_f(R_1) \subset \overline{\mathbb{D}}_{R_1} \subset \overline{\mathbb{D}}_1$, f is analytic in a neighborhood of ρ. We are given that $f(0) \neq 0$. This implies that the function f is not identically zero on any open connected set containing the origin. Since f is analytic on a connected open neighborhood of $\overline{\mathbb{D}}_1$, if f were identically zero on any open subset of its domain, it would have to be identically zero on the entire connected component, which would contradict $f(0) \neq 0$. Therefore, f is not identically zero in any neighborhood of ρ (this is the consequence of theorem 282). The quantity $m_{\rho,f}$ is defined as the analytic order of f at ρ. For a function that is analytic at a point ρ but not identically zero in a neighborhood of ρ, the order of a zero at ρ is a well-defined non-negative integer. Specifically, the order is the smallest integer $n \geq 0$ such that the n-th derivative $f^{(n)}(\rho)$ is non-zero. Since f is not identically zero around ρ, not all derivatives can be zero. Thus, $m_{\rho,f}$ must be a non-negative integer, i.e., $m_{\rho,f} \in \mathbb{N} = \{0, 1, 2, \dots\}$. □

Lemma 252 (Order at least one)**.** *Let $0 < R < 1$, $R_1 = \frac{2}{3}R$, and $f : \mathbb{C} \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$. If $f(0) \neq 0$ then $m_{\rho,f} \geq 1$ for all $\rho \in \mathcal{K}_f(R_1)$.*

Proof. Let ρ be an arbitrary element of $\mathcal{K}_f(R_1)$. From theorem 251, we have established that $m_{\rho,f}$ is a non-negative integer. The analytic order of a function f at a point ρ, $m_{\rho,f}$, is equal to 0 if and only if $f(\rho) \neq 0$. By the definition of the set of zeros $\mathcal{K}_f(R_1)$ (see theorem 280), for any $\rho \in \mathcal{K}_f(R_1)$, we have $f(\rho) = 0$. Since $f(\rho) = 0$, the order $m_{\rho,f}$ cannot be 0. Given that $m_{\rho,f}$ is a non-negative integer, it must be strictly greater than 0. Therefore, $m_{\rho,f} \geq 1$ for all $\rho \in \mathcal{K}_f(R_1)$. □

Lemma 253 (Analytic division)**.** *Let $D \subset \mathbb{C}$ be an open set, and let $w \in D$. Let $h : D \to \mathbb{C}$ and $g : D \to \mathbb{C}$ be functions that are analyticAt w. If $g(w) \neq 0$, then the function $z \mapsto h(z)/g(z)$ is analyticAt w.*

Proof. We are given that the functions h and g are analytic at w. This means they are complex differentiable in a neighborhood of w. We are also given the crucial assumption that $g(w) \neq 0$. Since g is analytic at w, it is also continuous at w. By the definition of continuity, for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $|z - w| < \delta$, then $|g(z) - g(w)| < \epsilon$. Let's choose $\epsilon = |g(w)|/2$. Since $g(w) \neq 0$, $\epsilon > 0$. Then there exists a neighborhood of w, say $U = D(w, \delta)$, such that for all $z \in U$, $|g(z) - g(w)| < |g(w)|/2$. This implies $g(z) \neq 0$ for all $z \in U$. Now consider the function $q(z) = 1/g(z)$ defined on the neighborhood U. The function $z \mapsto 1/z$ is analytic on $\mathbb{C} \setminus \{0\}$. Since g is analytic at w and its image $g(z)$ for $z \in U$ is contained in $\mathbb{C} \setminus \{0\}$, the composition $1/g$ is analytic at w. The function we are interested in is $h(z)/g(z)$, which can be written as the product of two functions: $h(z)$ and $q(z) = 1/g(z)$. The product of two functions that are analytic at a point w is also analytic at w. Since both $h(z)$ and $1/g(z)$ are analytic at w, their product $h(z)/g(z)$ is also analytic at w. ☐

Lemma 254 (Denominator analytic)**.** *Let $S \subset \mathbb{C}$ be a finite set, and for each $s \in S$, let $n_s \in \mathbb{N}$ be a positive integer. Then for all $w \notin S$, the function $P(z) = \prod_{s \in S}(z - s)^{n_s}$ is analyticAt w and $P(w) \neq 0$.*

Proof. Let w be an arbitrary point in $\mathbb{C} \setminus S$. First, we show that $P(z)$ is analytic at w. For each $s \in S$, consider the factor $f_s(z) = (z - s)^{n_s}$. This is a polynomial in z, and all polynomials are analytic on the entire complex plane $\mathbb{C}$. Therefore, each function $f_s(z)$ is analytic at w. The function $P(z)$ is defined as the product of the functions $f_s(z)$ for all s in the finite set S. A finite product of functions that are analytic at a point w is itself analytic at w. Therefore, $P(z)$ is analytic at w.

Next, we show that $P(w) \neq 0$. The value of the function at w is given by $P(w) = \prod_{s \in S}(w - s)^{n_s}$. A product of complex numbers is zero if and only if at least one of the factors is zero. Let's examine an arbitrary factor $(w - s)^{n_s}$ for some $s \in S$. We are given the assumption that $w \notin S$. This means that for any $s \in S$, we have $w \neq s$, which implies $w - s \neq 0$. Since n_s is a positive integer, $(w - s)^{n_s}$ is also non-zero. As this holds for every $s \in S$, none of the factors in the product $P(w)$ are zero. Therefore, the product $P(w)$ is not zero. ☐

Lemma 255 (Ratio analytic)**.** *Let $w \in \mathbb{C}$, $0 < R_1 < R < 1$, and $h : \overline{\mathbb{D}}_R \to \mathbb{C}$ be a function that is AnalyticAt w. Let $S \subset \overline{\mathbb{D}}_{R_1}$ be a finite set, and for each $s \in S$, let $n_s \in \mathbb{N}$ be a positive integer. Then for all $w \in \overline{\mathbb{D}}_1 \setminus S$, the function $h(z)/\prod_{s \in S}(z - s)^{n_s}$ is analyticAt w.*

Proof. Let $F(z) = \frac{h(z)}{\prod_{s \in S}(z-s)^{n_s}}$. We want to show that $F(z)$ is analytic at an arbitrary point $w \in \overline{\mathbb{D}}_1 \setminus S$. Let's define the denominator as $g(z) = \prod_{s \in S}(z - s)^{n_s}$. Then $F(z) = h(z)/g(z)$. We will use theorem 253. To do so, we must verify its hypotheses for the point w: 1. $h(z)$ is analytic at w. This is given as an assumption in the lemma statement. 2. $g(z)$ is analytic at w. 3. $g(w) \neq 0$.

We can verify the second and third hypotheses using theorem 254. The function $g(z)$ has the precise form required by theorem 254, with the set of roots being S and the exponents being n_s. The assumptions of theorem 254 are: a. S is a finite set. This is given in the current lemma's assumptions. b. For each $s \in S$, n_s is a positive integer. This is also given. c. The point of evaluation w is not in S. Our assumption is $w \in \overline{\mathbb{D}}_1 \setminus S$, which explicitly states $w \notin S$.

Since all assumptions of theorem 254 are met, we can conclude that the function $g(z)$ is analytic at w and that $g(w) \neq 0$. Now we have verified all three hypotheses for theorem 253. Therefore, we can conclude that the ratio $F(z) = h(z)/g(z)$ is analytic at w. ☐

Lemma 256 (Zero factorization)**.** *Let $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. For each $\sigma \in \mathcal{K}_f(R_1)$, there exists a function $h_\sigma(z)$ that is AnalyticAt σ, and $h_\sigma(\sigma) \neq 0$, and $f(z) = (z - \sigma)^{m_{\sigma,f}} h_\sigma(z)$ Eventually for z in nbhds of σ.*

Proof. Let σ be an arbitrary zero in $\mathcal{K}_f(R_1)$. By assumption, f is 'AnalyticOnNhd' $\overline{\mathbb{D}}_1$. This means there is an open neighborhood U of σ where f is analytic. On this neighborhood, f can be represented by its Taylor series centered at σ: $f(z) = \sum_{n=0}^{\infty} a_n(z-\sigma)^n$, where $a_n = \frac{f^{(n)}(\sigma)}{n!}$. Let $m = m_{\sigma,f}$. By theorem 250, m is the analytic order of f at σ. By definition of analytic order, this means that m is the smallest non-negative integer such that $f^{(m)}(\sigma) \neq 0$. Since $\sigma \in \mathcal{K}_f(R_1)$, we have $f(\sigma) = 0$. This implies that $m \geq 1$. The definition of order m implies that $f^{(k)}(\sigma) = 0$ for all integers $0 \leq k < m$, and $f^{(m)}(\sigma) \neq 0$. Consequently, the Taylor coefficients $a_k = f^{(k)}(\sigma)/k!$ are zero for $k < m$, and $a_m = f^{(m)}(\sigma)/m! \neq 0$. The Taylor series for $f(z)$ can thus be written as: $f(z) = a_m(z-\sigma)^m + a_{m+1}(z-\sigma)^{m+1} + a_{m+2}(z-\sigma)^{m+2} + \ldots$ We can factor out the term $(z-\sigma)^m$ from the series: $f(z) = (z-\sigma)^m \left(a_m + a_{m+1}(z-\sigma) + a_{m+2}(z-\sigma)^2 + \ldots\right)$. This holds for all z in the disk of convergence of the Taylor series, which is a neighborhood of σ. Let us define the function $h_\sigma(z)$ as the series in the parenthesis: $h_\sigma(z) = \sum_{j=0}^{\infty} a_{m+j}(z-\sigma)^j$. A power series defines an analytic function within its radius of convergence. This series for $h_\sigma(z)$ has the same radius of convergence as the series for $f(z)$, so $h_\sigma(z)$ is analytic in a neighborhood of σ, i.e., it is 'AnalyticAt' σ. By our construction, the identity $f(z) = (z-\sigma)^m h_\sigma(z)$ holds in this neighborhood. Finally, we must verify that $h_\sigma(\sigma) \neq 0$. We evaluate $h_\sigma(z)$ at $z = \sigma$: $h_\sigma(\sigma) = a_m + a_{m+1}(\sigma-\sigma) + a_{m+2}(\sigma-\sigma)^2 + \cdots = a_m$. As we established that $a_m \neq 0$, we have $h_\sigma(\sigma) \neq 0$. This completes the proof. □

Definition 257 (C function)**.** Let $0 < R_1 < R < 1$, and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. We define the function $C_f : \overline{\mathbb{D}}_R \to \mathbb{C}$ as follows. This function is constructed by dividing $f(z)$ by a polynomial whose roots are the zeros of f inside $\overline{\mathbb{D}}_{R_1}$. The definition is split into two cases to handle the points where the denominator would otherwise be zero.

$$C_f(z) = \begin{cases} \dfrac{f(z)}{\prod_{\rho \in \mathcal{K}_f(R_1)} (z-\rho)^{m_{\rho,f}}} & \text{if } z \neq \rho \text{ for all } \rho \in \mathcal{K}_f(R_1) \\[2ex] \dfrac{h_\sigma(\sigma)}{\prod_{\rho \in \mathcal{K}_f(R_1)\setminus\{\sigma\}} (\sigma-\rho)^{m_{\rho,f}}} & \text{if } z = \sigma \text{ for some } \sigma \in \mathcal{K}_f(R_1) \end{cases}$$

Lemma 258 (C analytic off K)**.** *Let $0 < R_1 < R < 1$, and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. Then $C_f(z)$ is analyticAt z for all $z \in \overline{\mathbb{D}}_R \setminus \mathcal{K}_f(R_1)$.*

Proof. Let z be an arbitrary point in the set $\overline{\mathbb{D}}_R \setminus \mathcal{K}_f(R_1)$. By the definition of this set, $z \notin \mathcal{K}_f(R_1)$. According to theorem 257, for such a point z, the function $C_f(z)$ is defined by the first case: $C_f(z) = \frac{f(z)}{\prod_{\rho \in \mathcal{K}_f(R_1)}(z-\rho)^{m_{\rho,f}}}$. We can prove this function is analytic at z by applying theorem 255. Let's verify its hypotheses:

- Let $h(z) = f(z)$, $S = \mathcal{K}_f(R_1)$, and for each $\rho \in S$, let $n_\rho = m_{\rho,f}$. The point of evaluation is $w = z$.
- The function $h(z) = f(z)$ is 'AnalyticOnNhd' $\overline{\mathbb{D}}_1$, so it is analytic at $z \in \overline{\mathbb{D}}_R \subset \overline{\mathbb{D}}_1$.
- The set $S = \mathcal{K}_f(R_1)$ is the set of zeros of a non-zero analytic function in a compact set $\overline{\mathbb{D}}_{R_1}$, and is therefore a finite set, as stated by theorem 249.
- For each $\rho \in S$, the exponent $n_\rho = m_{\rho,f}$ is a positive integer by theorem 252.
- The point of evaluation $w = z$ is in $\overline{\mathbb{D}}_R \setminus S$, which is a subset of $\overline{\mathbb{D}}_1 \setminus S$.

All hypotheses of theorem 255 are satisfied. Therefore, we conclude that $C_f(z)$ is analytic at z. Since z was an arbitrary point in $\overline{\mathbb{D}}_R \setminus \mathcal{K}_f(R_1)$, the statement holds for all such points. $\square$

Lemma 259 (C at zero)**.** *Let $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. For each $\sigma \in \mathcal{K}_f(R_1)$, we have Eventually for z in nbhds of σ, if $z = \sigma$ then*

$$C_f(z) = \frac{h_\sigma(z)}{\prod_{\rho \in \mathcal{K}_f(R_1)\setminus\{\sigma\}} (z-\rho)^{m_{\rho,f}}}.$$

Proof. Let σ be an arbitrary point in $\mathcal{K}_f(R_1)$. We are interested in the case where $z = \sigma$. By theorem 257, when $z = \sigma$, $C_f(z)$ is defined by the second case: $C_f(\sigma) = \frac{h_\sigma(\sigma)}{\prod_{\rho \in \mathcal{K}_f(R_1)\setminus\{\sigma\}} (\sigma-\rho)^{m_{\rho,f}}}$. The expression we are asked to prove is $C_f(z) = \frac{h_\sigma(z)}{\prod_{\rho \in \mathcal{K}_f(R_1)\setminus\{\sigma\}} (z-\rho)^{m_{\rho,f}}}$. Evaluating the right-hand side at $z = \sigma$ gives: $\frac{h_\sigma(\sigma)}{\prod_{\rho \in \mathcal{K}_f(R_1)\setminus\{\sigma\}} (\sigma-\rho)^{m_{\rho,f}}}$. This is precisely the definition of $C_f(\sigma)$. Thus, the equality holds at $z = \sigma$. The phrase "Eventually for z in nbhds of σ" is satisfied trivially, as the statement only concerns the point $z = \sigma$ itself and holds true at that point regardless of the neighborhood. $\square$

Lemma 260 (Zeros isolated)**.** *Let $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. For any $\sigma, \rho \in \mathcal{K}_f(R_1)$ with $\sigma \neq \rho$, Eventually for z in nbhds of σ, we have $z \neq \rho$.*

Proof. The statement "Eventually for z in nbhds of σ, we have $z \neq \rho$" means that there exists a neighborhood of σ that does not contain ρ. Let σ and ρ be two distinct points in $\mathcal{K}_f(R_1)$. Let $d = |\sigma - \rho|$ be the distance between them. Since $\sigma \neq \rho$, we have $d > 0$. Consider the open disk $U = D(\sigma, d)$ centered at σ with radius d. This is a neighborhood of σ. For any point $z \in U$, the distance from z to σ is less than d, i.e., $|z - \sigma| < d$. The distance from any such z to ρ is $|z - \rho|$. By the reverse triangle inequality, $|z-\rho| = |(z-\sigma)-(\rho-\sigma)| \geq ||\rho-\sigma|-|z-\sigma|| = |d-|z-\sigma||$. Since $|z - \sigma| < d$, the value $d - |z - \sigma|$ is positive. Thus, $|z - \rho| > 0$, which implies $z \neq \rho$. Therefore, the neighborhood U of σ does not contain the point ρ. This proves the claim. $\square$

Lemma 261 (C near zero)**.** *Let $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. For each $\sigma \in \mathcal{K}_f(R_1)$, we have Eventually for z in nbhds of σ, if $z \neq \sigma$ then*

$$C_f(z) = \frac{(z-\sigma)^{m_{\sigma,f}} h_\sigma(z)}{\prod_{\rho \in \mathcal{K}_f(R_1)} (z-\rho)^{m_{\rho,f}}}.$$

Proof. Let σ be an arbitrary point in $\mathcal{K}_f(R_1)$. By theorem 249, the set $\mathcal{K}_f(R_1)$ is finite. Let $\mathcal{K}_f(R_1) \setminus \{\sigma\} = \{\rho_1, \rho_2, \ldots, \rho_k\}$. For each ρ_i in this set, since $\rho_i \neq \sigma$, by theorem 260 there exists a neighborhood U_i of σ such that for all $z \in U_i$, $z \neq \rho_i$. Let $U = \bigcap_{i=1}^k U_i$. As a finite intersection of neighborhoods of σ, U is also a neighborhood of σ. For any $z \in U$, we have $z \neq \rho_i$ for all $i = 1, \ldots, k$. Now, consider a point z in this neighborhood U such that $z \neq \sigma$. For such a z, we have $z \notin \{\rho_1, \ldots, \rho_k\}$ and $z \neq \sigma$, which means $z \notin \mathcal{K}_f(R_1)$. By the first case of theorem 257, for such a z, we have $C_f(z) = \frac{f(z)}{\prod_{\rho \in \mathcal{K}_f(R_1)} (z-\rho)^{m_{\rho,f}}}$. From theorem 256, there exists a neighborhood of σ, say V, and a function $h_\sigma(z)$ such that $f(z) = (z-\sigma)^{m_{\sigma,f}} h_\sigma(z)$ for all $z \in V$. Let $W = U \cap V$. This is also a neighborhood of σ. For any $z \in W$ with $z \neq \sigma$, both of the above representations are valid. Substituting the expression for $f(z)$ into the one for $C_f(z)$, we get: $C_f(z) = \frac{(z-\sigma)^{m_{\sigma,f}} h_\sigma(z)}{\prod_{\rho \in \mathcal{K}_f(R_1)} (z-\rho)^{m_{\rho,f}}}$. This holds for all z in the punctured neighborhood $W \setminus \{\sigma\}$, which satisfies the "Eventually" condition. $\square$

Lemma 262 (Product split)**.** *Let* $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ *be a function that is AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) \neq 0$*. For each* $\sigma \in \mathcal{K}_f(R_1)$ *we have*

$$\prod_{\rho \in \mathcal{K}_f(R_1)} (z-\rho)^{m_{\rho,f}} = (z-\sigma)^{m_{\sigma,f}} \prod_{\rho \in \mathcal{K}_f(R_1)\setminus\{\sigma\}} (z-\rho)^{m_{\rho,f}}$$

Proof. Let $S = \mathcal{K}_f(R_1)$. By theorem 249, S is a finite set. Let σ be any element of S. The set S can be partitioned into two disjoint subsets: $\{\sigma\}$ and $S \setminus \{\sigma\}$. The product over the finite set S can be split into the product of the terms corresponding to these two subsets. Let $a_\rho(z) = (z-\rho)^{m_{\rho,f}}$. The product over S is $\prod_{\rho \in S} a_\rho(z)$. By the commutative property of multiplication, we can separate the term for $\rho = \sigma$: $\prod_{\rho \in S} a_\rho(z) = a_\sigma(z) \cdot \left(\prod_{\rho \in S\setminus\{\sigma\}} a_\rho(z)\right)$. Substituting the definition of $a_\rho(z)$ back into this identity gives: $\prod_{\rho \in \mathcal{K}_f(R_1)} (z-\rho)^{m_{\rho,f}} = (z-\sigma)^{m_{\sigma,f}} \cdot \left(\prod_{\rho \in \mathcal{K}_f(R_1)\setminus\{\sigma\}} (z-\rho)^{m_{\rho,f}}\right)$. This is a fundamental property of products over finite sets. □

Lemma 263 (Product quotient)**.** *Let* $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ *be a function that is AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) \neq 0$*. For each* $\sigma \in \mathcal{K}_f(R_1)$ *and* $z \notin \mathcal{K}_f(R_1)$*, we have*

$$\frac{(z-\sigma)^{m_{\sigma,f}}}{\prod_{\rho \in \mathcal{K}_f(R_1)} (z-\rho)^{m_{\rho,f}}} = \frac{1}{\prod_{\rho \in \mathcal{K}_f(R_1)\setminus\{\sigma\}} (z-\rho)^{m_{\rho,f}}}$$

Proof. From theorem 262, we have the identity: $\prod_{\rho \in \mathcal{K}_f(R_1)} (z-\rho)^{m_{\rho,f}} = (z-\sigma)^{m_{\sigma,f}} \prod_{\rho \in \mathcal{K}_f(R_1)\setminus\{\sigma\}} (z-\rho)^{m_{\rho,f}}$. To manipulate this equation by division, we must ensure the terms we divide by are non-zero. The crucial assumption is that $z \notin \mathcal{K}_f(R_1)$. This means that for every $\rho \in \mathcal{K}_f(R_1)$, we have $z \neq \rho$, and therefore $z - \rho \neq 0$. Since $m_{\rho,f} \geq 1$, it follows that $(z-\rho)^{m_{\rho,f}} \neq 0$ for all $\rho \in \mathcal{K}_f(R_1)$. This implies that all factors in the products are non-zero. In particular, the denominator on the left-hand side, $\prod_{\rho \in \mathcal{K}_f(R_1)} (z-\rho)^{m_{\rho,f}}$, is non-zero. Also, the term $(z-\sigma)^{m_{\sigma,f}}$ is non-zero. We can therefore divide both sides of the identity from theorem 262 by the non-zero quantity $(z-\sigma)^{m_{\sigma,f}} \cdot \left(\prod_{\rho \in \mathcal{K}_f(R_1)} (z-\rho)^{m_{\rho,f}}\right)$. Starting with the identity and dividing by $\prod_{\rho \in \mathcal{K}_f(R_1)} (z-\rho)^{m_{\rho,f}}$ gives: $1 = \frac{(z-\sigma)^{m_{\sigma,f}} \prod_{\rho \in \mathcal{K}_f(R_1)\setminus\{\sigma\}} (z-\rho)^{m_{\rho,f}}}{\prod_{\rho \in \mathcal{K}_f(R_1)} (z-\rho)^{m_{\rho,f}}}$. Now, dividing by the non-zero term $\prod_{\rho \in \mathcal{K}_f(R_1)\setminus\{\sigma\}} (z-\rho)^{m_{\rho,f}}$ yields the desired result: $\frac{1}{\prod_{\rho \in \mathcal{K}_f(R_1)\setminus\{\sigma\}} (z-\rho)^{m_{\rho,f}}} = \frac{(z-\sigma)^{m_{\sigma,f}}}{\prod_{\rho \in \mathcal{K}_f(R_1)} (z-\rho)^{m_{\rho,f}}}$. □

Lemma 264 (C off K)**.** *Let* $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ *be a function that is AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) \neq 0$*. For each* $\sigma \in \mathcal{K}_f(R_1)$*, we have Eventually for* z *in nbhds of* σ*, if* $z \neq \sigma$ *then*

$$C_f(z) = \frac{h_\sigma(z)}{\prod_{\rho \in \mathcal{K}_f(R_1)\setminus\{\sigma\}} (z-\rho)^{m_{\rho,f}}}.$$

Proof. By theorem 261, there exists a neighborhood W of σ such that for all $z \in W$ with $z \neq \sigma$, we have the identity: $C_f(z) = \frac{(z-\sigma)^{m_{\sigma,f}} h_\sigma(z)}{\prod_{\rho \in \mathcal{K}_f(R_1)} (z-\rho)^{m_{\rho,f}}}$. We can rewrite the right-hand side as a product: $C_f(z) = h_\sigma(z) \cdot \left(\frac{(z-\sigma)^{m_{\sigma,f}}}{\prod_{\rho \in \mathcal{K}_f(R_1)} (z-\rho)^{m_{\rho,f}}}\right)$. For a point $z \in W$ with $z \neq \sigma$, we established in the proof of theorem 261 that $z \notin \mathcal{K}_f(R_1)$. Therefore, the conditions for theorem 263 are met for such a point z. Applying this lemma, we can replace the fractional part: $\frac{(z-\sigma)^{m_{\sigma,f}}}{\prod_{\rho\in\mathcal{K}_f(R_1)}(z-\rho)^{m_{\rho,f}}} = \frac{1}{\prod_{\rho\in\mathcal{K}_f(R_1)\setminus\{\sigma\}}(z-\rho)^{m_{\rho,f}}}$. Substituting this back into the expression for $C_f(z)$ gives: $C_f(z) = h_\sigma(z)\cdot \frac{1}{\prod_{\rho\in\mathcal{K}_f(R_1)\setminus\{\sigma\}}(z-\rho)^{m_{\rho,f}}} = \frac{h_\sigma(z)}{\prod_{\rho\in\mathcal{K}_f(R_1)\setminus\{\sigma\}}(z-\rho)^{m_{\rho,f}}}$. This equality holds for all $z \in W\setminus\{\sigma\}$, which satisfies the "Eventually" condition. □

Lemma 265 (C local form). *Let $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0)\neq 0$. For each $\sigma\in\mathcal{K}_f(R_1)$, we have Eventually for z in nbhds of σ,*

$$C_f(z) = \frac{h_\sigma(z)}{\prod_{\rho\in\mathcal{K}_f(R_1)\setminus\{\sigma\}}(z-\rho)^{m_{\rho,f}}}.$$

Proof. Let σ be an arbitrary point in $\mathcal{K}_f(R_1)$. Let us define the function $g_\sigma(z) = \frac{h_\sigma(z)}{\prod_{\rho\in\mathcal{K}_f(R_1)\setminus\{\sigma\}}(z-\rho)^{m_{\rho,f}}}$. From theorem 264, we know there exists a neighborhood of σ, let's call it W, such that for all $z\in W\setminus\{\sigma\}$, the equality $C_f(z) = g_\sigma(z)$ holds. From theorem 259, we know that at the point $z=\sigma$, the equality $C_f(\sigma) = g_\sigma(\sigma)$ also holds. Combining these two results, we see that $C_f(z) = g_\sigma(z)$ for all points z in the neighborhood W. This proves the statement. □

Lemma 266 (h ratio analytic). *Let $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0)\neq 0$. For each $\sigma\in\mathcal{K}_f(R_1)$, the function $\frac{h_\sigma(z)}{\prod_{\rho\in\mathcal{K}_f(R_1)\setminus\{\sigma\}}(z-\rho)^{m_{\rho,f}}}$ is analyticAt σ.*

Proof. Let σ be an arbitrary point in $\mathcal{K}_f(R_1)$. We want to prove that the function $g_\sigma(z) = \frac{h_\sigma(z)}{\prod_{\rho\in\mathcal{K}_f(R_1)\setminus\{\sigma\}}(z-\rho)^{m_{\rho,f}}}$ is analytic at σ. We will use theorem 255 with the point of evaluation $w=\sigma$. Let's identify the components and verify the hypotheses:

- The numerator function is $h(z) = h_\sigma(z)$.
- The set of roots in the denominator is $S = \mathcal{K}_f(R_1)\setminus\{\sigma\}$.
- The exponents are $n_\rho = m_{\rho,f}$ for each $\rho\in S$.

Now we check the conditions of theorem 255:

1. $h(z)$ must be analytic at σ. By theorem 256, the function $h_\sigma(z)$ is analytic at σ. This condition is met.
2. S must be a finite set. Since $\mathcal{K}_f(R_1)$ is finite (by theorem 249), its subset S is also finite. This condition is met.
3. For each $\rho\in S$, n_ρ must be a positive integer. For $\rho\in S$, $n_\rho = m_{\rho,f}$. By theorem 252, $m_{\rho,f}\geq 1$. This condition is met.
4. The point of evaluation σ must not be in S. By definition, $S = \mathcal{K}_f(R_1)\setminus\{\sigma\}$, so $\sigma\notin S$. This condition is met.

Since all hypotheses of theorem 255 are satisfied, we can conclude that the function $g_\sigma(z)$ is analytic at σ. □

Lemma 267 (C analytic at K). *Let $0 < R_1 < R < 1$, and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0)\neq 0$. Then for every $\sigma\in\mathcal{K}_f(R_1)$, the function $C_f(z)$ is analyticAt σ.*

Proof. Let σ be an arbitrary point in $\mathcal{K}_f(R_1)$. By theorem 265, there exists a neighborhood of σ, say W, such that for all $z \in W$, $C_f(z)$ is equal to the function $g_\sigma(z) = \frac{h_\sigma(z)}{\prod_{\rho \in \mathcal{K}_f(R_1)\setminus\{\sigma\}} (z-\rho)^{m_{\rho,f}}}$. By theorem 266, the function $g_\sigma(z)$ is analytic at σ. A function is defined to be analytic at a point σ if it is equal to a function known to be analytic at σ in a neighborhood of σ. Since $C_f(z) = g_\sigma(z)$ on the neighborhood W and $g_\sigma(z)$ is analytic at σ, it follows directly that $C_f(z)$ is also analytic at σ. As σ was an arbitrary element of $\mathcal{K}_f(R_1)$, this holds for all points in that set. □

Lemma 268 (C is analytic)**.** *Let $0 < R_1 < R < 1$, and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. Then $C_f(z)$ is analyticAt z for all $z \in \overline{\mathbb{D}}_R$.*

Proof. Let z be an arbitrary point in the closed disk $\overline{\mathbb{D}}_R$. We must show that C_f is analytic at z. We can partition the domain $\overline{\mathbb{D}}_R$ into two disjoint sets: those points that are in $\mathcal{K}_f(R_1)$ and those that are not. Note that $\mathcal{K}_f(R_1) \subset \overline{\mathbb{D}}_{R_1} \subset \overline{\mathbb{D}}_R$. **Case 1:** The point z is not in $\mathcal{K}_f(R_1)$. In this case, $z \in \overline{\mathbb{D}}_R \setminus \mathcal{K}_f(R_1)$. By theorem 258, the function C_f is analytic at z. **Case 2:** The point z is in $\mathcal{K}_f(R_1)$. In this case, let's call the point $\sigma = z$. By theorem 267, the function C_f is analytic at σ. Since any point $z \in \overline{\mathbb{D}}_R$ must fall into one of these two cases, and we have shown that C_f is analytic at z in both cases, we conclude that $C_f(z)$ is analytic for all $z \in \overline{\mathbb{D}}_R$. □

Lemma 269 (f nonzero off K)**.** *Let $0 < R_1 < R < 1$, and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. Then $f(z) \neq 0$ for all $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$.*

Proof. We prove this by contraposition. The contrapositive statement is: if $z \in \overline{\mathbb{D}}_{R_1}$ and $f(z) = 0$, then $z \in \mathcal{K}_f(R_1)$. Let z be a point in $\overline{\mathbb{D}}_{R_1}$ such that $f(z) = 0$. The set $\mathcal{K}_f(R_1)$ is defined (in theorem 240) as the set of all points w in the closed disk $\overline{\mathbb{D}}_{R_1}$ for which $f(w) = 0$. Since $z \in \overline{\mathbb{D}}_{R_1}$ and $f(z) = 0$, z satisfies the condition for membership in $\mathcal{K}_f(R_1)$. Therefore, $z \in \mathcal{K}_f(R_1)$. This proves the contrapositive, and thus the original statement is true. □

Lemma 270 (C nonzero off K)**.** *Let $0 < R_1 < R < 1$, and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. Then $C_f(z) \neq 0$ for all $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$.*

Proof. Let z be an arbitrary point in the set $\overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$. By definition, $z \notin \mathcal{K}_f(R_1)$. According to the first case of theorem 257, $C_f(z)$ is given by the ratio: $C_f(z) = \frac{f(z)}{\prod_{\rho \in \mathcal{K}_f(R_1)} (z-\rho)^{m_{\rho,f}}}$. A fraction is non-zero if and only if its numerator is non-zero and its denominator is finite and non-zero. **Numerator:** The numerator is $f(z)$. Since $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$, by theorem 269, we have $f(z) \neq 0$. **Denominator:** The denominator is the product $P(z) = \prod_{\rho \in \mathcal{K}_f(R_1)} (z-\rho)^{m_{\rho,f}}$. Since $\mathcal{K}_f(R_1)$ is finite, this is a finite product. For the product to be non-zero, each of its factors must be non-zero. A factor is of the form $(z-\rho)^{m_{\rho,f}}$. Since we assumed $z \notin \mathcal{K}_f(R_1)$, we have $z \neq \rho$ for all $\rho \in \mathcal{K}_f(R_1)$. This means $z - \rho \neq 0$. As $m_{\rho,f} \geq 1$, it follows that $(z-\rho)^{m_{\rho,f}} \neq 0$. Since every factor is non-zero, the denominator is non-zero. Since the numerator is non-zero and the denominator is non-zero, their ratio $C_f(z)$ must be non-zero. □

Lemma 271 (C nonzero on K)**.** *Let $0 < R_1 < R < 1$, and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. Then $C_f(\sigma) \neq 0$ for all $\sigma \in \mathcal{K}_f(R_1)$.*

Proof. Let σ be an arbitrary point in $\mathcal{K}_f(R_1)$. By the second case of theorem 257, the value of C_f at σ is given by: $C_f(\sigma) = \frac{h_\sigma(\sigma)}{\prod_{\rho \in \mathcal{K}_f(R_1)\setminus\{\sigma\}} (\sigma-\rho)^{m_{\rho,f}}}$. We must show this expression is non-zero. **Numerator:** The numerator is $h_\sigma(\sigma)$. By theorem 256, the function h_σ is constructed specifically to satisfy $h_\sigma(\sigma) \neq 0$. **Denominator:** The denominator is the product $\prod_{\rho \in \mathcal{K}_f(R_1)\setminus\{\sigma\}} (\sigma - \rho)^{m_{\rho,f}}$. This is a finite product. For any ρ in the indexing set $\mathcal{K}_f(R_1)\setminus\{\sigma\}$, we have $\rho \neq \sigma$, which implies $\sigma - \rho \neq 0$. Since $m_{\rho,f} \geq 1$, the factor $(\sigma-\rho)^{m_{\rho,f}}$ is also non-zero. As a finite product of non-zero terms, the denominator is non-zero. Since the numerator is non-zero and the denominator is non-zero, their ratio $C_f(\sigma)$ is non-zero. □

Lemma 272 (C never zero)**.** *Let $0 < R_1 < R < 1$, and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. Then $C_f(z) \neq 0$ for all $z \in \overline{\mathbb{D}}_{R_1}$.*

Proof. Let z be an arbitrary point in the closed disk $\overline{\mathbb{D}}_{R_1}$. We partition the domain $\overline{\mathbb{D}}_{R_1}$ into two disjoint sets: $\mathcal{K}_f(R_1)$ and $\overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$. Any point $z \in \overline{\mathbb{D}}_{R_1}$ must belong to exactly one of these sets. **Case 1:** $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$. By theorem 270, we have $C_f(z) \neq 0$. **Case 2:** $z \in \mathcal{K}_f(R_1)$. By theorem 271, we have $C_f(z) \neq 0$. In both possible cases, $C_f(z)$ is non-zero. Therefore, we conclude that $C_f(z) \neq 0$ for all $z \in \overline{\mathbb{D}}_{R_1}$. □

Lemma 273 (Blaschke diff)**.** *Let $0 < R_1 < R < 1$ and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) = 1$. Then the function $z \mapsto \prod_{\rho \in \mathcal{K}_f(R_1)} (R - \bar{\rho}z/R)^{m_{\rho,f}}$ is differentiableAt z for all $z \in \overline{\mathbb{D}}_R$.*

Proof. By theorem 354. Note $\mathcal{K}_f(R_1)$ is finite by theorem 249 □

Lemma 274 (Blaschke nonzero)**.** *Let $0 < R_1 < R < 1$ and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) = 1$. Then $\prod_{\rho \in \mathcal{K}_f(R_1)} (R - \bar{\rho}z/R)^{m_{\rho,f}} \neq 0$ for all $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$.*

Proof. By theorem 354. Note $\mathcal{K}_f(R_1)$ is finite by theorem 249 □

Definition 275 (Blaschke B)**.** Let $0 < R_1 < R < 1$, and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. Define the function $B_f : \overline{\mathbb{D}}_R \to \mathbb{C}$ as $B_f(z) = C_f(z) \prod_{\rho \in \mathcal{K}_f(R_1)} (R - \bar{\rho}z/R)^{m_{\rho,f}}$.

Lemma 276 (B and C relation)**.** *Let $0 < R_1 < R < 1$, and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. For $z \in \overline{\mathbb{D}}_R \setminus \mathcal{K}_f(R_1)$, we have*

$$B_f(z) = f(z) \frac{\prod_{\rho \in \mathcal{K}_f(R_1)} (R - \bar{\rho}z/R)^{m_{\rho,f}}}{\prod_{\rho \in \mathcal{K}_f(R_1)} (z - \rho)^{m_{\rho,f}}}$$

Proof. By theorems 257 and 275 □

Lemma 277 (B division)**.** *Let $0 < R_1 < R < 1$, and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. For $z \in \overline{\mathbb{D}}_R \setminus \mathcal{K}_f(R_1)$, we have*

$$\frac{\prod_{\rho \in \mathcal{K}_f(R_1)} (R - \bar{\rho}z/R)^{m_{\rho,f}}}{\prod_{\rho \in \mathcal{K}_f(R_1)} (z - \rho)^{m_{\rho,f}}} = \prod_{\rho \in \mathcal{K}_f(R_1)} \frac{(R - \bar{\rho}z/R)^{m_{\rho,f}}}{(z - \rho)^{m_{\rho,f}}}$$

Proof. By Mathlib: Finset.prod_div_distrib Note $\mathcal{K}_f(R_1)$ is finite by theorem 249. □

Lemma 278 (B product pow)**.** *Let $0 < R_1 < R < 1$, and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. For $z \in \overline{\mathbb{D}}_R \setminus \mathcal{K}_f(R_1)$, we have*

$$\prod_{\rho \in \mathcal{K}_f(R_1)} \frac{(R - \bar{\rho}z/R)^{m_{\rho,f}}}{(z - \rho)^{m_{\rho,f}}} = \prod_{\rho \in \mathcal{K}_f(R_1)} \left(\frac{R - \bar{\rho}z/R}{z - \rho}\right)^{m_{\rho,f}}$$

Proof. By theorem 277 and Mathlib: div_pow Note $m_{\rho,f} \in \mathbb{N}$. Note $\mathcal{K}_f(R_1)$ is finite by theorem 249. □

Lemma 279 (B off K)**.** *Let $0 < R_1 < R < 1$, and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. For $z \in \overline{\mathbb{D}}_R \setminus \mathcal{K}_f(R_1)$, we have*

$$B_f(z) = f(z) \prod_{\rho \in \mathcal{K}_f(R_1)} \left(\frac{R - \bar{\rho} z/R}{z - \rho} \right)^{m_{\rho,f}}.$$

Proof. By theorem 276 and theorems 277 and 278 □

2.2 Bounding $K \leq 3 \log B$

Lemma 280 (Zero value)**.** *Let $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$. If $\rho \in \mathcal{K}_f(R_1)$ then $f(\rho) = 0$.*

Proof. Unfold definition 240. □

Lemma 281 (Zero contrapositive)**.** *Let $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$. If $f(\rho) \neq 0$ then $\rho \notin \mathcal{K}_f(R_1)$.*

Proof. Contrapositive of theorem 280 □

Lemma 282 (Not zero)**.** *Let $f : \mathbb{C} \to \mathbb{C}$. If $f(0) \neq 0$, then f is not the identically zero function.*

Proof. By definition of the identically zero function. □

Lemma 283 (Disk bound)**.** *Let $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$. If $\rho \in \mathcal{K}_f(R_1)$ then $|\rho| \leq R_1$.*

Proof. By theorem 240, as $\mathcal{K}_f(R_1)$ is a subset of $\overline{\mathbb{D}}_{R_1}$. □

Lemma 284 (Zero excluded)**.** *Let $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$. If $f(0) \neq 0$ then $0 \notin \mathcal{K}_f(R_1)$.*

Proof. By theorem 281. □

Lemma 285 (Nonzero rho)**.** *Let $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$. If $f(0) \neq 0$ then $\rho \neq 0$ for all $\rho \in \mathcal{K}_f(R_1)$.*

Proof. By theorem 284, as ρ is an element of $\mathcal{K}_f(R_1)$. □

Lemma 286 (Mod positive)**.** *Let $z \in \mathbb{C}$. If $z \neq 0$ then $|z| > 0$.*

Proof. Shown in theorem 35 □

Lemma 287 (Rho positive)**.** *Let $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$. If $f(0) \neq 0$ then $|\rho| > 0$ for all $\rho \in \mathcal{K}_f(R_1)$.*

Proof. By theorem 285 and theorem 286. □

Lemma 288 (Disk bound)**.** *Let $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$. If $\rho \in \mathcal{K}_f(R_1)$ then $|\rho| \leq R_1$.*

Proof. By theorem 240, as $\mathcal{K}_f(R_1)$ is a subset of $\overline{\mathbb{D}}_{R_1}$. □

Lemma 289 (Inverse mono)**.** *Let $x, y \in \mathbb{R}$. If $0 < x \leq y$, then $1/x \geq 1/y$.*

Proof. □

Lemma 290 (Inverse bound). *Let $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ with $f(0) \neq 0$. If $\rho \in \mathcal{K}_f(R_1)$ then $1/|\rho| \geq 1/R_1$.*

Proof. By theorem 287, theorem 288, and theorem 289. □

Lemma 291 (Mult inequality). *Let $a, b, c \in \mathbb{R}$. If $a \leq b$ and $c > 0$, then $ac \leq bc$.*

Proof. □

Lemma 292 (Ratio bound). *Let $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ with $f(0) \neq 0$. If $\rho \in \mathcal{K}_f(R_1)$ then $R/|\rho| \geq R/R_1$.*

Proof. By theorem 290 and theorem 291, using the hypothesis $R > 0$. □

Lemma 293 (Mod product). *Let $\{w_\rho\}_{\rho \in K}$ be a finite collection of complex numbers. We have $|\prod_{\rho \in I} w_\rho| = \prod_{\rho \in K} |w_\rho|$.*

Proof. □

Lemma 294 (B modulus). *Let $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$. If $\mathcal{K}_f(R_1)$ if finite, then $z \in \overline{\mathbb{D}}_R \setminus \mathcal{K}_f(R_1)$, we have*

$$|B_f(z)| = |f(z)| \prod_{\rho \in \mathcal{K}_f(R_1)} \left| \left(\frac{R - z\bar{\rho}/R}{z - \rho} \right)^{m_\rho} \right|$$

.

Proof. By theorem 279 and theorem 293 with $I = \mathcal{K}_f(R_1)$ and $w_\rho = (\frac{R - z\bar{\rho}/R}{z - \rho})^{m_\rho}$. □

Lemma 295 (Abs power). *For $n \in \mathbb{N}$ and $w \in \mathbb{C}$, we have $|w^n| = |w|^n$.*

Proof. □

Lemma 296 (Power mod). *Let $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. Then $\left| \left(\frac{R - z\bar{\rho}/R}{z - \rho} \right)^{m_\rho} \right| = \left| \frac{R - z\bar{\rho}/R}{z - \rho} \right|^{m_\rho}$.*

Proof. By theorem 295 with $n = m_\rho$ and $w = \frac{R - z\bar{\rho}/R}{z - \rho}$. □

Lemma 297 (B modulus). *Let $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$. If $\mathcal{K}_f(R_1)$ if finite, then $|B_f(z)| = |f(z)| \prod_{\rho \in \mathcal{K}_f(R_1)} \left| \frac{R - z\bar{\rho}/R}{z - \rho} \right|^{m_\rho}$.*

Proof. By theorems 294 and 296. □

Lemma 298 (B at zero). *Let $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. Then $|B_f(0)| = |f(0)| \prod_{\rho \in \mathcal{K}_f(R_1)} \left| \frac{R}{-\rho} \right|^{m_\rho} |$.*

Proof. By evaluating the expression in theorem 294 at $z = 0$. □

Lemma 299 (Abs division). *Let $w_1, w_2 \in \mathbb{C}$ with $w_2 \neq 0$. We have $|w_1/w_2| = |w_1|/|w_2|$.*

Proof. □

Lemma 300 (Abs neg). *Let $w \in \mathbb{C}$. We have $|-w| = |w|$.*

Proof. ☐

Lemma 301 (Abs ratio)**.** *Let $R > 0$ and $\rho \in \mathbb{C}$ with $\rho \neq 0$. We have $|\frac{R}{-\rho}| = |R|/|\rho|$.*

Proof. By theorems 285, 299 and 300 with $w_1 = R$ and ρ. ☐

Lemma 302 (B zero form)**.** *Let $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. Then $|B_f(0)| = |f(0)| \prod_{\rho \in \mathcal{K}_f(R_1)} (|R|/|\rho|)^{m_\rho}$.*

Proof. By applying theorems 285 and 301 to the expression in theorem 298. ☐

Lemma 303 (Abs positive)**.** *Let $x \in \mathbb{R}$. If $x > 0$, then $|x| = x$.*

Proof. ☐

Lemma 304 (B zero form)**.** *Let $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. Then $|B_f(0)| = |f(0)| \prod_{\rho \in \mathcal{K}_f(R_1)} (R/|\rho|)^{m_\rho}$.*

Proof. By applying theorem 303 with $x = R > 0$ to the expression in theorem 302. ☐

Lemma 305 (Prod inequality)**.** *Let K be a finite set, $a : K \to \mathbb{R}$, and $b : K \to \mathbb{R}$. If $0 \leq a_\rho \leq b_\rho$ for all $\rho \in K$, then $\prod_{\rho \in K} a_\rho \leq \prod_{\rho \in K} b_\rho$.*

Proof. ☐

Lemma 306 (Power bound)**.** *Let $n \in \mathbb{N}$. If $c > 1$ and $n \geq 1$, then $c \leq c^n$.*

Proof. ☐

Lemma 307 (Power one)**.** *Let $n \in \mathbb{N}$. If $c \geq 1$ and $n \geq 1$, then $1 \leq c^n$.*

Proof. Apply theorem 306, and then assumption $1 \leq c$. ☐

Lemma 308 (Product power)**.** *Let K be a finite set. If $c_\rho \geq 1$, $n_\rho \in \mathbb{N}$, and $n_\rho \geq 1$ for all $\rho \in K$, then $\prod_{\rho \in K} c_\rho^{n_\rho} \geq \prod_{\rho \in K} 1$.*

Proof. By theorem 305 $c_\rho \leq c_\rho^{n_\rho}$. Then apply theorem 306 with $a_\rho = c_\rho$, $b_\rho = c_\rho^{n_\rho}$. ☐

Lemma 309 (Product one)**.** *Let K be a finite set. Then $\prod_{\rho \in K} 1 = 1$.*

Proof. ☐

Lemma 310 (Power bound)**.** *Let K be a finite set. If $c_\rho \geq 1$, $n_\rho \in \mathbb{N}$, and $n_\rho \geq 1$ for all $\rho \in K$, then $\prod_{\rho \in K} c_\rho^{n_\rho} \geq 1$.*

Proof. By theorems 308 and 309. ☐

Lemma 311 (Modulus bound)**.** *Let $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) = 1$. Then $\prod_{\rho \in \mathcal{K}_f(R_1)} (3/2)^{m_\rho} \geq 1$.*

Proof. First $m_\rho \in \mathbb{N}$ and $m_\rho \geq 1$ by theorems 251 and 252. Also $\mathcal{K}_f(R_1)$ is finite by theorem 249. Now apply theorem 310 with $K = \mathcal{K}_f(R_1)$, $b_\rho = 3/2$, and $n = m_\rho$. ☐

Lemma 312 (B analytic)**.** *Let $0 < R_1 < R < 1$ and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) \neq 0$. Then $B_f(z)$ is AnalyticOnNhd $\overline{\mathbb{D}}_R$.*

Proof. By theorem 268. ☐

Lemma 313 (Boundary mod)**.** *Let $R > 0$ and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$. Then $|B_f(z)| = |f(z)|$ for all $|z| = R$.*

Proof. ☐

Lemma 314 (Boundary bound)**.** *Let $B > 1$, $0 < R < 1$ and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$. If $|f(z)| \le B$ on $|z| \le R$ then $|B_f(z)| \le B$ for all $|z| = R$.*

Proof. By theorem 313 and the hypothesis $|f(z)| \le B$. ☐

Lemma 315 (Max modulus)**.** *Let $B > 1$, $0 < R < 1$ and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$. If $B_f(z)$ is AnalyticOnNhd $\overline{\mathbb{D}}_R$ and $|B_f(z)| \le B$ for all $|z| = R$, then $|B_f(z)| \le B$ for all $|z| \le R$.*

Proof. By applying theorem 110 with $h(z) = B_f(z)$. ☐

Lemma 316 (Disk bound)**.** *Let $B > 1$, $0 < R < 1$ and $f : \mathbb{C} \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$. If $|B_f(z)| \le B$ for all $|z| = R$, then $|B_f(z)| \le B$ for all $|z| \le R$.*

Proof. By theorem 312 and theorem 315. ☐

Lemma 317 (Disk bound)**.** *Let $B > 1$, $0 < R < 1$ and $f : \mathbb{C} \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$. If $|f(z)| \le B$ on $|z| \le R$, then $|B_f(z)| \le B$ for all $|z| \le R$.*

Proof. By theorem 314 and theorem 316. ☐

Lemma 318 (Zero bound)**.** *Let $B > 1$, $0 < R < 1$ and $f : \mathbb{C} \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$. If $|f(z)| \le B$ on $|z| \le R$, then $|B_f(0)| \le B$.*

Proof. By evaluating the inequality in theorem 317 at $z = 0$. ☐

Lemma 319 (Combine bounds)**.** *Let $B > 1$, $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) = 1$. If $|B_f(0)| \le B$ then $(3/2)^{\sum_{\rho \in \mathcal{K}_f(R_1)} m_\rho} \le B$.*

Proof. ☐

Lemma 320 (Jensen form)**.** *Let $B > 1$, $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$. If $f(0) = 1$ and $|f(z)| \le B$ on $|z| \le R$, then $(3/2)^{\sum_{\rho \in \mathcal{K}_f(R_1)} m_\rho} \le B$.*

Proof. By theorem 318 and theorem 319. ☐

Lemma 321 (Log monotone)**.** *Let $x, y \in \mathbb{R}$. If $0 < x \le y$, then $\log x \le \log y$.*

Proof. ☐

Lemma 322 (Jensen log)**.** *Let $B > 1$, $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$. If $f(0) = 1$ and $|f(z)| \le B$ on $|z| \le R$, then $\sum_{\rho \in \mathcal{K}_f(R_1)} m_\rho \log(3/2) \le \log B$.*

Proof. By applying theorem 321 to the inequality in theorem 320. ☐

Lemma 323 (Three exceeds e)**.** *We have $3 > \exp(1)$.*

Proof. ☐

Lemma 324 (Multiplicity bound)**.** *Let $B > 1$, $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$. If $f(0) = 1$ and $|f(z)| \le B$ on $|z| \le R$, then $\sum_{\rho \in \mathcal{K}_f(R_1)} m_\rho \le \frac{\log B}{\log(R/R_1)}$.*

Proof. Note $\log(R/R_1) > 0$ since $R/R_1 > 1$. By theorem 322, and then divide both sides by $\log(R/R_1)$. ☐

Lemma 325 (Sum inequality)**.** *Let K be a finite set, $a : K \to \mathbb{R}$, and $b : K \to \mathbb{R}$. If $a_\rho \leq b_\rho$ for all $\rho \in K$, then $\sum_{\rho\in K} a_\rho \leq \sum_{\rho\in K} b_\rho$.*

Proof. ☐

Lemma 326 (Multiplicity one)**.** *Let $B > 1$, $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$. Then $\sum_{\rho\in\mathcal{K}_f(R_1)} 1 \leq \sum_{\rho\in\mathcal{K}_f(R_1)} m_\rho$.*

Proof. $\mathcal{K}_f(R_1)$ is finite by theorem 249. Now apply theorems 252 and 325 with $K = \mathcal{K}_f(R_1)$, $a_\rho = 1$, and $b_\rho = m_\rho$. ☐

Lemma 327 (Ones bound)**.** *Let $B > 1$, $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$. If $f(0) = 1$ and $|f(z)| \leq B$ on $|z| \leq R$, then $\sum_{\rho\in\mathcal{K}_f(R_1)} 1 \leq \frac{\log B}{\log(R/R_1)}$.*

Proof. By theorems 324 and 326. ☐

Lemma 328 (Count identity)**.** *Let S be a finite set. Then $\sum_{s\in S} 1 = |S|$.*

Proof. ☐

Lemma 329 (Zeros bound)**.** *Let $B > 1$, $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$. If $f(0) = 1$ and $|f(z)| \leq B$ on $|z| \leq R$, then $|\mathcal{K}_f(R_1)| \leq \frac{\log B}{\log(R/R_1)}$.*

Proof. $\mathcal{K}_f(R_1)$ is finite by theorem 249. Now apply theorems 327 and 328 with $S = \mathcal{K}_f(R_1)$ ☐

2.3 Log L_f

Definition 330 (Log function)**.** Let $0 < R < 1$, $B > 1$, and $f : \mathbb{C} \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) = 1$. Define $L_f(z) = J_{B_f}(z)$ from theorem 238, where B_f from theorem 275.

Lemma 331 (Disk analytic)**.** *Let $0 < R_1 < R < 1$ and $f : \mathbb{C} \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$. Then $B_f(z)$ is AnalyticOnNhd $\overline{\mathbb{D}}_R$.*

Proof. By theorems 268 and 273 ☐

Lemma 332 (Never zero)**.** *Let $0 < R_1 < R < 1$ and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$. Then $B_f(z) \neq 0$ for all $z \in \overline{\mathbb{D}}_{R_1}$.*

Proof. By theorems 272 and 274 ☐

Lemma 333 (B zero)**.** *Let $0 < R_1 < R < 1$ and $f : \mathbb{C} \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$. Then $B_f(0) \neq 0$.*

Proof. By theorem 332 with $z = 0$. ☐

Lemma 334 (Lf analytic)**.** *Let $0 < R_1 < R < 1$ and $f : \mathbb{C} \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) = 1$. Then $L_f(z)$ is AnalyticOnNhd $\overline{\mathbb{D}}_r$.*

Proof. By theorems 238 and 330. ☐

Lemma 335 (Lf at zero)**.** *Let* $0 < R_1 < R < 1$ *and* $f : \mathbb{C} \to \mathbb{C}$ *be a function that is AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. We have* $L_f(0) = 0$*.*

Proof. By theorems 238 and 330. □

Lemma 336 (Real part diff)**.** *Let* $0 < R_1 < R < 1$ *and* $f : \mathbb{C} \to \mathbb{C}$ *be a function that is AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. Then* $\Re(L_f(z)) = \log|B_f(z)| - \log|B_f(0)|$ *on* $\overline{\mathbb{D}}_r$*.*

Proof. By theorems 238 and 330 and theorem 237 □

Lemma 337 (Log bound)**.** *Let* $B > 1$*,* $0 < R_1 < R < 1$ *and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$*. If* $0 < |B_f(z)|$ *and* $|B_f(z)| \leq B$ *for all* $|z| \leq R_1$*, then* $\log|B_f(z)| \leq \log B$ *for all* $|z| \leq R_1$*.*

Proof. By theorem 321 with $x = |B_f(z)|$ and $y = B$. □

Lemma 338 (Log bound)**.** *Let* $B > 1$*,* $0 < R_1 < R < 1$ *and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$*. If* $|B_f(z)| \leq B$ *for all* $|z| \leq R$*, then* $\log|B_f(z)| \leq \log B$ *for all* $|z| \leq R_1$*.*

Proof. By theorems 332 and 337. □

Lemma 339 (Log bound)**.** *Let* $B > 1$*,* $0 < R_1 < R < 1$ *and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$*. If* $|f(z)| \leq B$ *on* $|z| \leq R$*, then* $\log|B_f(z)| \leq \log B$ *for all* $|z| \leq R_1$*.*

Proof. By theorems 317 and 338. □

Lemma 340 (Log nonnegative)**.** *Let* $0 < R_1 < R < 1$ *and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. Then* $\log|B_f(0)| \geq 0$*.*

Proof. By theorem 321 with $x = 1$ and $y = |B_f(0)|$, giving $\log|B_f(0)| \geq \log(1) = 0$. □

Lemma 341 (Real part bound)**.** *Let* $B > 1$*,* $0 < r < R_1 < R < 1$ *and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$*. If* $f(0) = 1$ *and* $|f(z)| \leq B$ *on* $|z| \leq R$*, then* $\Re(L_f(z)) \leq \log B$ *for all* $|z| \leq r$*.*

Proof. By theorems 336, 339 and 340. □

Lemma 342 (BC inequality)**.** *Let* $M > 0$ *and* $0 < r_1 < r < 1$*. Let* L *be analytic on* $|z| \leq r$ *such that* $L(0) = 0$ *and suppose* $\Re(L(z)) \leq M$ *for all* $|z| \leq r$*. Then for any* $|z| \leq r_1$*,*

$$|L'(z)| \leq \frac{16Mr^2}{(r - r_1)^3}.$$

Proof. By theorem 198. □

Lemma 343 (Apply BC)**.** *Let* $B > 1$*,* $0 < r_1 < r < R_1 < R < 1$ *and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$*. If* $f(0) = 1$ *and* $|f(z)| \leq B$ *on* $|z| \leq R$*. For any* $|z| \leq r_1$

$$|L_f'(z)| \leq \frac{16\log(B)r^2}{(r - r_1)^3}.$$

Proof. By theorem 342 with $r := r$, $r_1 := r_1$, $L(z) = L_f(z)$ and $M = \log B$, using theorems 334, 335 and 341. □

2.4 Log derivative L_f' expansion

Lemma 344 (Constant rule). *Let $a \in \mathbb{C}$ with $a \neq 0$ and $g : \overline{D}_1 \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$. Then logDeriv$(a \cdot g(z))$ = logDeriv$(g(z))$.*

Proof. Mathlib: logDeriv_const_mul □

Lemma 345 (One minus B). *Let $0 < R < 1$ and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) = 1$. Then $B_f(0) \neq 0$ and $1/B_f(0) \neq 0$.*

Proof. By theorem 331 and theorem 333 □

Lemma 346 (Derivative form). *Let $0 < R < 1$ and $f : \overline{\mathbb{D}}_1 \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) = 1$. We have logDeriv$(B_f(z)/B_f(0))$ = logDeriv$(B_f(z))$.*

Proof. By theorems 344 and 345 with $g(z) = B_f(z)$ and $a = 1/B_f(0)$. □

Lemma 347 (Product rule). *Let f, g be functions differentiableAt z with $f(z), g(z) \neq 0$. Then logDeriv$(f \cdot g)$ = logDeriv(f) + logDeriv(g) at z.*

Proof. Mathlib: logDeriv_mul □

Lemma 348 (Product sum). *Let K be a finite set and $\{g_\rho(z)\}_{\rho \in K}$ be a collection of functions differentiableAt z with $g_\rho(z) \neq 0$ for all $\rho \in K$. Then logDeriv$(\prod_{\rho \in K} g_\rho(z)) = \sum_{\rho \in K}$ logDeriv$(g_\rho(z))$.*

Proof. Mathlib: logDeriv_prod □

Lemma 349 (Quotient rule). *Let h, g be functions differentiableAt z with $h(z), g(z) \neq 0$. Then logDeriv(h/g) = logDeriv(h) − logDeriv(g) at z.*

Proof. Mathlib: logDeriv_div □

Lemma 350 (Power rule). *Let $m \in \mathbb{N}$ and let g be a function differentiableAt z. Then logDeriv$(g(z)^m) = m \cdot$ logDeriv$(g(z))$.*

Proof. Mathlib: logDeriv_fun_pow □

Lemma 351 (Difference nonzero). *Let $0 < R < 1$, $R_1 = \frac{2}{3}R$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) = 1$. Then for any $\rho \in \mathcal{K}_f(R_1)$, the function $z \mapsto z - \rho$ is never equal to zero, and differentiableAt z for all $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$.*

Proof. By definition $z \notin \mathcal{K}_f(R_1)$ and $\rho \in \mathcal{K}_f(R_1)$. This implies that $z \neq \rho$, and therefore $z - \rho \neq 0$. The function $z \mapsto z - \rho$ is a linear function, therefore differentiableAt z. □

Lemma 352 (Numerator nonzero). *Let $0 < R < 1$, $R_1 = \frac{2}{3}R$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{D}_{R_1}$ with $f(0) = 1$. Then for any $\rho \in \mathcal{K}_f(R_1)$, the function $z \mapsto R - \bar{\rho}z/R$ is never equal to zero, and differentiableAt z for all $z \in \overline{\mathbb{D}}_1$.*

Proof. $R - \bar{\rho}z/R \neq 0$ for all $z \in \overline{D}_R \setminus \mathcal{K}_f(R_1)$.
The function $R - \bar{\rho}z/R$ is a linear function, therefore differentiableAt z. □

Lemma 353 (Fraction nonzero). *Let $0 < R < 1$, $R_1 = \frac{2}{3}R$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) = 1$. Then for any $\rho \in \mathcal{K}_f(R_1)$, the function $z \mapsto \frac{R - \bar{\rho}z/R}{z - \rho}$ is never equal to zero, and differentiableAt z for all $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$.*

Proof. By theorems 351 and 352 □

Lemma 354 (Power nonzero)**.** *Let* $0 < R < 1$*,* $R_1 = \frac{2}{3}R$*, and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. Then for any* $\rho \in \mathcal{K}_f(R_1)$*, the function* $z \mapsto (\frac{R-\bar{\rho}z/R}{z-\rho})^{m_{\rho,f}}$ *is never equal to zero, and differentiableAt* z *for all* $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$*.*

Proof. By theorem 353. Note $m_{\rho,f} \in \mathbb{N}$. □

Lemma 355 (Product nonzero)**.** *Let* $0 < R < 1$*,* $R_1 = \frac{2}{3}R$*, and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. Then the function* $z \mapsto \prod_{\rho\in\mathcal{K}_f(R_1)}(\frac{R-\bar{\rho}z/R}{z-\rho})^{m_{\rho,f}}$ *is never equal to zero, and differentiableAt* z *for all* $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$*.*

Proof. By theorem 354. Note $\mathcal{K}_f(R_1)$ is finite by theorem 249 □

Lemma 356 (Outside zeros)**.** *Let* $0 < R < 1$*,* $R_1 = \frac{2}{3}R$*, and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. Then the function* $f(z)$ *is never equal to zero, and differentiableAt* z *for all* $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$*.*

Proof. By definition of $\mathcal{K}_f(R_1)$ in theorem 240. □

Lemma 357 (Outside zeros)**.** *Let* $0 < R < 1$*,* $R_1 = \frac{2}{3}R$*, and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. Then the function* $B_f(z)$ *is never equal to zero, and differentiableAt* z *for all* $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$*.*

Proof. By theorems 355 and 356, recalling theorem 275. □

Lemma 358 (Log sum)**.** *Let* $0 < R < 1$*,* $R_1 = \frac{2}{3}R$*, and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. Then for all* $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$ *we have*

$$\mathit{logDeriv}\left(f(z)\prod_{\rho\in\mathcal{K}_f(R_1)}\left(\frac{R-z\bar{\rho}/R}{z-\rho}\right)^{m_{\rho,f}}\right) = \mathit{logDeriv}(f(z)) + \mathit{logDeriv}\left(\prod_{\rho\in\mathcal{K}_f(R_1)}\left(\frac{R-z\bar{\rho}/R}{z-\rho}\right)^{m_{\rho,f}}\right).$$

Proof. By theorem 347 with $g(z) = \prod_{\rho\in\mathcal{K}_f(R_1)}\left(\frac{R-z\bar{\rho}/R}{z-\rho}\right)^{m_{\rho,f}}$.

Note diffAt, nonzero conditions hold by theorem 355. □

Lemma 359 (Log sum)**.** *Let* $0 < R < 1$*,* $R_1 = \frac{2}{3}R$*, and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. Then for all* $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$ *we have*

$$\mathit{logDeriv}(B_f(z)) = \mathit{logDeriv}(f(z)) + \mathit{logDeriv}\left(\prod_{\rho\in\mathcal{K}_f(R_1)}\left(\frac{R-z\bar{\rho}/R}{z-\rho}\right)^{m_{\rho,f}}\right).$$

Proof. By theorems 275 and 358. □

Lemma 360 (Fraction form)**.** *Let* f *be a non-zero analytic function. Then* $\mathit{logDeriv}(f(z)) = \frac{f'}{f}(z)$*.*

Proof. By definition of logDeriv in Mathlib. □

Lemma 361 (Step one)**.** *Let* $0 < r < R_1 < R < 1$*, and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. Then for all* $z \in \overline{\mathbb{D}}_r \setminus \mathcal{K}_f(R_1)$ *we have*

$$L'_f(z) = \frac{f'}{f}(z) + logDeriv\left(\prod_{\rho \in \mathcal{K}_f(R_1)} \left(\frac{R - z\bar{\rho}/R}{z - \rho}\right)^{m_{\rho,f}}\right)$$

Proof. By theorem 346, theorem 359, and theorem 360. □

Lemma 362 (Product sum)**.** *Let* $0 < R < 1$*,* $R_1 = \frac{2}{3}R$*, and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. Then for all* $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$ *we have*

$$logDeriv\left(\prod_{\rho \in \mathcal{K}_f(R_1)} \left(\frac{R - z\bar{\rho}/R}{z - \rho}\right)^{m_{\rho,f}}\right) = \sum_{\rho \in \mathcal{K}_f(R_1)} logDeriv\left(\left(\frac{R - z\bar{\rho}/R}{z - \rho}\right)^{m_{\rho,f}}\right)$$

Proof. By theorem 348 with $K = \mathcal{K}_f(R_1)$ and $g_\rho(z) = \left(\frac{R - z\bar{\rho}/R}{z - \rho}\right)^{m_{\rho,f}}$.

Note $K = \mathcal{K}_f(R_1)$ is finite by theorem 249. The diffAt, nonzero conditions hold by theorem 354. □

Lemma 363 (Power multiple)**.** *Let* $0 < R < 1$*,* $R_1 = \frac{2}{3}R$*, and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. Then for all* $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$ *and* $\rho \in \mathcal{K}_f(R_1)$ *we have*

$$logDeriv\left(\left(\frac{R - z\bar{\rho}/R}{z - \rho}\right)^{m_{\rho,f}}\right) = m_{\rho,f}\, logDeriv\left(\frac{R - z\bar{\rho}/R}{z - \rho}\right).$$

Proof. By theorem 350 with $m = m_{\rho,f}$ and $g(z) = \frac{R - z\bar{\rho}/R}{z - \rho}$. Note $m_{\rho,f} \in \mathbb{N}$. The diffAt, nonzero conditions hold by theorem 353. □

Lemma 364 (Sum multiple)**.** *Let* $0 < R < 1$*,* $R_1 = \frac{2}{3}R$*, and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. Then for all* $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$ *we have*

$$logDeriv\left(\prod_{\rho \in \mathcal{K}_f(R_1)} \left(\frac{R - z\bar{\rho}/R}{z - \rho}\right)^{m_{\rho,f}}\right) = \sum_{\rho \in \mathcal{K}_f(R_1)} m_{\rho,f}\, logDeriv\left(\frac{R - z\bar{\rho}/R}{z - \rho}\right).$$

Proof. By theorem 362 and theorem 363. □

Lemma 365 (Step two)**.** *Let* $0 < r < R_1 < R$*,* $R < 1$*, and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. Then for all* $z \in \overline{\mathbb{D}}_r \setminus \mathcal{K}_f(R_1)$ *we have*

$$L'_f(z) = \frac{f'}{f}(z) + \sum_{\rho \in \mathcal{K}_f(R_1)} m_{\rho,f}\, logDeriv\left(\frac{R - z\bar{\rho}/R}{z - \rho}\right).$$

Proof. By theorem 361 and theorem 364. □

Lemma 366 (Difference form)**.** *Let* $0 < R < 1$*,* $R_1 = \frac{2}{3}R$*, and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. Then for all* $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$ *and* $\rho \in \mathcal{K}_f(R_1)$ *we have*

$$logDeriv\left(\frac{R - z\bar{\rho}/R}{z - \rho}\right) = logDeriv(R - z\bar{\rho}/R) - logDeriv(z - \rho).$$

Proof. By theorem 349 with $h(z) = R - z\bar{\rho}/R$ and $g(z) = z - \rho$.

Note diffAt, nonzero conditions hold by theorems 351 and 352. □

Lemma 367 (Linear rule)**.** *Let $a, b \in \mathbb{C}$ with $a \neq 0$. We have logDeriv$(az + b) = \frac{a}{az+b}$ at $z \neq -b/a$.*

Proof. Note linear polynomial has derivative $(az + b)' = a$.

Now unfold logDeriv definition and calculate logDeriv$(az + b) = \frac{(az+b)'}{az+b} = \frac{a}{az+b}$. □

Lemma 368 (Denominator rule)**.** *Let $\rho \in \mathbb{C}$. We have logDeriv$(z - \rho) = \frac{1}{z-\rho}$ at $z \neq \rho$.*

Proof. By theorem 367 with $a = 1$ and $b = -\rho$. □

Lemma 369 (Numerator rule)**.** *Let $R, \rho \in \mathbb{C}$. We have logDeriv$(R - z\bar{\rho}/R) = \frac{-\bar{\rho}/R}{R - z\bar{\rho}/R}$.*

Proof. By theorem 367 with $a = -\bar{\rho}/R$ and $b = R$. □

Lemma 370 (Rearranged form)**.** *Let $R, \rho \in \mathbb{C}$. We have $\frac{-\bar{\rho}/R}{R - z\bar{\rho}/R} = \frac{1}{z - R^2/\bar{\rho}}$.*

Proof. This is an algebraic simplification. For the expression to be well-defined, we must make some assumptions that are implicit in the context of the larger proof:

- $R \neq 0$ and $\bar{\rho} \neq 0$ (which implies $\rho \neq 0$), so that the fractions are defined.
- The denominator $R - z\bar{\rho}/R \neq 0$.
- The denominator $z - R^2/\bar{\rho} \neq 0$.

These conditions hold in the domains where this lemma is applied.

Our goal is to show the equality of the two fractions. We will start with the left-hand side (LHS) and manipulate it to obtain the right-hand side (RHS). The strategy is to multiply the numerator and the denominator of the LHS by the same non-zero quantity, chosen to simplify the expression. A suitable choice is the factor $-R/\bar{\rho}$.

Let's start with the LHS:

$$\text{LHS} = \frac{-\bar{\rho}/R}{R - z\bar{\rho}/R}$$

Now, we multiply the numerator and the denominator by $-R/\bar{\rho}$:

$$\text{LHS} = \frac{(-\bar{\rho}/R) \cdot (-R/\bar{\rho})}{(R - z\bar{\rho}/R) \cdot (-R/\bar{\rho})}$$

Let's simplify the new numerator and denominator separately.

Numerator simplification:

$$(-\bar{\rho}/R) \cdot (-R/\bar{\rho}) = \frac{-\bar{\rho}}{R} \cdot \frac{-R}{\bar{\rho}} = \frac{(-\bar{\rho})(-R)}{R\bar{\rho}} = \frac{\bar{\rho}R}{R\bar{\rho}} = 1$$

Denominator simplification: We distribute the factor $(-R/\bar{\rho})$ across the terms in the denominator:

$$\begin{aligned}(R - z\bar{\rho}/R) \cdot (-R/\bar{\rho}) &= R \cdot (-R/\bar{\rho}) - (z\bar{\rho}/R) \cdot (-R/\bar{\rho}) \\ &= -\frac{R^2}{\bar{\rho}} - \left(\frac{z\bar{\rho}}{R} \cdot \frac{-R}{\bar{\rho}}\right) \\ &= -\frac{R^2}{\bar{\rho}} - \left(z \cdot \frac{\bar{\rho}(-R)}{R\bar{\rho}}\right) \\ &= -\frac{R^2}{\bar{\rho}} - (z \cdot (-1)) \\ &= -\frac{R^2}{\bar{\rho}} + z \\ &= z - \frac{R^2}{\bar{\rho}}\end{aligned}$$

Conclusion: Substituting the simplified numerator and denominator back into the fraction, we get:

$$\text{LHS} = \frac{1}{z - R^2/\bar{\rho}}$$

This is exactly the RHS of the equation we wanted to prove. □

Lemma 371 (Numerator form)**.** *Let $R, \rho \in \mathbb{C}$. We have logDeriv$(R - z\bar{\rho}/R) = \frac{1}{z-R^2/\bar{\rho}}$.*

Proof. By theorems 369 and 370 □

Lemma 372 (Diff fraction)**.** *Let $0 < R < 1$, $R_1 = \frac{2}{3}R$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$. For $\rho \in \mathcal{K}_f(R_1)$, we have logDeriv$\left(\frac{R-z\bar{\rho}/R}{z-\rho}\right) = \frac{1}{z-R^2/\bar{\rho}} - \frac{1}{z-\rho}$.*

Proof. The proof proceeds by first applying the division rule for the logarithmic derivative and then evaluating each resulting term. This is valid for any $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$ and any $\rho \in \mathcal{K}_f(R_1)$.

Step 1: Apply the division rule for logDeriv We use theorem 366, which is an application of the general rule $\text{logDeriv}(h/g) = \text{logDeriv}(h) - \text{logDeriv}(g)$. Let $h(z) = R - z\bar{\rho}/R$ and $g(z) = z - \rho$. To apply this rule, we must ensure that $h(z)$ and $g(z)$ are differentiable and non-zero at z.

- For $h(z) = R - z\bar{\rho}/R$: Theorem 352 confirms that this function is differentiable and non-zero for all $z \in \overline{\mathbb{D}}_1$, which includes our domain of interest.
- For $g(z) = z - \rho$: Theorem 351 confirms that this function is differentiable and non-zero for all $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$, given $\rho \in \mathcal{K}_f(R_1)$.

Since the conditions are met, we can apply theorem 366 to get:

$$\text{logDeriv}\left(\frac{R - z\bar{\rho}/R}{z - \rho}\right) = \text{logDeriv}(R - z\bar{\rho}/R) - \text{logDeriv}(z - \rho).$$

Step 2: Evaluate the first term, logDeriv$(R - z\bar{\rho}/R)$ We use theorem 371. This lemma states:

$$\text{logDeriv}(R - z\bar{\rho}/R) = \frac{1}{z - R^2/\bar{\rho}}.$$

This is valid provided $R \neq 0$ and $\rho \neq 0$, which are true under our assumptions ($0 < R < 1$ and $\rho \in \mathcal{K}_f(R_1)$ implies $\rho \neq 0$ as $f(0) = 1$).

Step 3: Evaluate the second term, logDeriv$(z - \rho)$ We use theorem 368. This lemma states:

$$\text{logDeriv}(z - \rho) = \frac{1}{z - \rho}.$$

This is valid for $z \neq \rho$. This condition is satisfied, as our domain for z is $\overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$ and ρ is an element of $\mathcal{K}_f(R_1)$, so z cannot be equal to ρ.

Step 4: Substitute the results back Now we substitute the expressions found in Step 2 and Step 3 into the equation from Step 1:

$$\text{logDeriv}\left(\frac{R - z\bar{\rho}/R}{z - \rho}\right) = \left(\frac{1}{z - R^2/\bar{\rho}}\right) - \left(\frac{1}{z - \rho}\right).$$

This gives the final desired formula. □

Lemma 373 (Step three)**.** *Let $0 < r < R_1 < R$, $R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) = 1$. Then for all $z \in \overline{\mathbb{D}}_r \setminus \mathcal{K}_f(R_1)$ we have $L'_f(z) = \frac{f'}{f}(z) + \sum_{\rho \in \mathcal{K}_f(R_1)} m_{\rho,f} \left(\frac{1}{z - R^2/\bar{\rho}} - \frac{1}{z - \rho}\right)$.*

Proof. The proof is a direct substitution into a previously established formula. The assumptions on R and f are used to justify the application of the necessary lemmas. The result holds for all $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$.

Step 1: Recall the formula for $L'_f(z)$ from theorem 365 Theorem 365 provides an expression for $L'_f(z)$ under the same assumptions as the current lemma. The formula is:

$$L'_f(z) = \frac{f'}{f}(z) + \sum_{\rho \in \mathcal{K}_f(R_1)} m_{\rho,f} \text{logDeriv}\left(\frac{R - z\bar{\rho}/R}{z - \rho}\right).$$

This equation holds for all $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$.

Step 2: Find a replacement for the logDeriv term Our goal is to replace the term $\text{logDeriv}\left(\frac{R - z\bar{\rho}/R}{z - \rho}\right)$ inside the summation. We look to theorem 372. This lemma gives the following identity for each $\rho \in \mathcal{K}_f(R_1)$ and for all $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$:

$$\text{logDeriv}\left(\frac{R - z\bar{\rho}/R}{z - \rho}\right) = \frac{1}{z - R^2/\bar{\rho}} - \frac{1}{z - \rho}.$$

Step 3: Substitute the expression into the formula for $L'_f(z)$ We now substitute the expression from Step 2 into the formula from Step 1. The substitution is valid because the domains for z and ρ match in both lemmas. Starting with the formula from Step 1:

$$L'_f(z) = \frac{f'}{f}(z) + \sum_{\rho \in \mathcal{K}_f(R_1)} m_{\rho,f} \left(\text{logDeriv}\left(\frac{R - z\bar{\rho}/R}{z - \rho}\right)\right).$$

We replace the parenthesized term with its equivalent from Step 2:

$$L'_f(z) = \frac{f'}{f}(z) + \sum_{\rho \in \mathcal{K}_f(R_1)} m_{\rho,f} \left(\frac{1}{z - R^2/\bar{\rho}} - \frac{1}{z - \rho}\right).$$

This is the final expression we aimed to prove. □

Lemma 374 (Sum difference). *Let K be a finite set and $a, b : K \to \mathbb{C}$. Then $\sum_{\rho \in K}(a_\rho - b_\rho) = \sum_{\rho \in K} a_\rho - \sum_{\rho \in K} b_\rho$.*

Proof. By the distributive property of summation. □

Lemma 375 (Sum rearranged). *Let $0 < R < 1$, $R_1 = \frac{2}{3}R$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) = 1$. Then for all $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$ we have*

$$\sum_{\rho \in \mathcal{K}_f(R_1)} m_{\rho,f} \left(\frac{1}{z - R^2/\bar{\rho}} - \frac{1}{z - \rho} \right) = \sum_{\rho \in \mathcal{K}_f(R_1)} \frac{m_{\rho,f}}{z - R^2/\bar{\rho}} - \sum_{\rho \in \mathcal{K}_f(R_1)} \frac{m_{\rho,f}}{z - \rho}.$$

Proof. By theorem 374. Note $K = \mathcal{K}_f(R_1)$ is finite by theorem 249 □

Lemma 376 (Final formula). *Let $0 < r < R_1 < R$, $R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) = 1$. Then for all $z \in \overline{\mathbb{D}}_r \setminus \mathcal{K}_f(R_1)$ we have*

$$L'_f(z) = \frac{f'}{f}(z) - \sum_{\rho \in \mathcal{K}_f(R_1)} \frac{m_{\rho,f}}{z - \rho} + \sum_{\rho \in \mathcal{K}_f(R_1)} \frac{m_{\rho,f}}{z - R^2/\bar{\rho}}.$$

Proof. By theorem 373 and theorem 375. □

Lemma 377 (Rearranged deriv). *Let $0 < r < R_1 < R$, $R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) = 1$. Then for all $z \in \overline{\mathbb{D}}_r \setminus \mathcal{K}_f(R_1)$ we have*

$$\frac{f'}{f}(z) - \sum_{\rho \in \mathcal{K}_f(R_1)} \frac{m_{\rho,f}}{z - \rho} = L'_f(z) - \sum_{\rho \in \mathcal{K}_f(R_1)} \frac{m_{\rho,f}}{z - R^2/\bar{\rho}}.$$

Proof. By algebraic rearrangement of the equality in theorem 376. □

Lemma 378 (Triangle sum). *Let $w_1, w_2 \in \mathbb{C}$. We have $|w_1 - w_2| \le |w_1| + |w_2|$.*

Proof. By the triangle inequality. □

Lemma 379 (Setup inequality). *Let $0 < r < R_1 < R$, $R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) = 1$. Then for all $z \in \overline{\mathbb{D}}_r \setminus \mathcal{K}_f(R_1)$ we have*

$$\left| \frac{f'}{f}(z) - \sum_{\rho \in \mathcal{K}_f(R_1)} \frac{m_{\rho,f}}{z - \rho} \right| \le |L'_f(z)| + \left| \sum_{\rho \in \mathcal{K}_f(R_1)} \frac{m_{\rho,f}}{z - R^2/\bar{\rho}} \right|.$$

Proof. By applying the modulus and theorem 378 to the equality in theorem 377. □

Lemma 380 (Step two). *Let $0 < R_1 < R < 1$ and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1$ with $f(0) = 1$. Then for all $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$ we have*

$$\sum_{\rho \in \mathcal{K}_f(R_1)} \frac{m_{\rho,f}}{|z - R^2/\bar{\rho}|} \le \frac{1}{R^2/R_1 - R_1} \sum_{\rho \in \mathcal{K}_f(R_1)} m_{\rho,f}.$$

Proof. Note $K = \mathcal{K}_f(R_1)$ is finite by theorem 249. □

Lemma 381 (Final sum)**.** *Let* $B > 1$, $0 < R_1 < R < 1$ *and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. If* $|f(z)| \le B$ *on* $|z| \le R$*, then for all* $z \in \overline{\mathbb{D}}_{R_1} \setminus \mathcal{K}_f(R_1)$ *we have*

$$\left| \sum_{\rho \in \mathcal{K}_f(R_1)} \frac{m_{\rho,f}}{z - R^2/\bar{\rho}} \right| \le \frac{\log B}{(R^2/R_1 - R_1)\log(R/R_1)}.$$

Proof. By theorem 380, and theorem 324. □

Lemma 382 (Final bound)**.** *Let* $B > 1$, $0 < r_1 < r < R_1 < R < 1$*, and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. If* $|f(z)| \le B$ *on* $|z| \le R$*, then for all* $z \in \overline{\mathbb{D}}_{r_1} \setminus \mathcal{K}_f(R_1)$ *we have*

$$\left| \frac{f'}{f}(z) - \sum_{\rho \in \mathcal{K}_f(R_1)} \frac{m_{\rho,f}}{z - \rho} \right| \le \frac{16\log(B)r^2}{(r - r_1)^3} + \frac{\log B}{(R^2/R_1 - R_1)\log(R/R_1)}.$$

Proof. By theorem 379, theorem 381, and theorem 343. □

Lemma 383 (Final bound)**.** *Let* $B > 1$, $0 < r_1 < r < R_1 < R < 1$*, and* $f : \mathbb{C} \to \mathbb{C}$ *AnalyticOnNhd* $\overline{\mathbb{D}}_1$ *with* $f(0) = 1$*. If* $|f(z)| \le B$ *on* $|z| \le R$*, then for all* $z \in \overline{\mathbb{D}}_{r_1} \setminus \mathcal{K}_f(R_1)$ *we have*

$$\left| \frac{f'}{f}(z) - \sum_{\rho \in \mathcal{K}_f(R_1)} \frac{m_{\rho,f}}{z - \rho} \right| \le \left(\frac{16r^2}{(r - r_1)^3} + \frac{1}{(R^2/R_1 - R_1)\log(R/R_1)} \right) \log B.$$

Proof. By theorem 382, factoring out $\log B$ from both terms. □

Chapter 3

Riemann Zeta Function

3.1 Zeta lower bound

Definition 384 (Prime set)**.** Let $\mathcal{P}$ be Nat.Primes

Lemma 385 (Prime decay)**.** *For $p \in \mathcal{P}$ and $s \in \mathbb{C}$ with $\Re(s) > 1$, we have $|p^{-s}| < 1$.*

Proof. Let $\sigma = \Re(s)$. By hypothesis, $\sigma > 1$. By Lemma 404, we have $|p^{-s}| = p^{-\sigma}$. Since $p \in \mathcal{P}$, we have $p \geq 2$. As $\sigma > 1$, it follows that $p^\sigma > p^1 \geq 2$. Therefore, $p^{-\sigma} = 1/p^\sigma < 1$. □

Lemma 386 (Euler product)**.** *For $s \in \mathbb{C}$ with $\Re(s) > 1$, the function $w_s(p) = (1 - p^{-s})^{-1}$ is multipliable, and we have $\zeta(s) = \prod'_{p \in \mathcal{P}} (1 - p^{-s})^{-1}$.*

Proof. Mathlib: riemannZeta_eulerProduct_hasProd, riemannZeta_eulerProduct_tprod, □

Lemma 387 (Abs product)**.** *Let P be a set and $w : P \to \mathbb{C}$ be multipliable. Then $|\prod'_{p \in P} w(p)| = \prod'_{p \in P} |w(p)|$.*

Proof. Mathlib: abs_tprod □

Lemma 388 (Abs primes)**.** *For $s \in \mathbb{C}$ with $\Re(s) > 1$, we have $|\prod'_{p \in \mathcal{P}} (1 - p^{-s})^{-1}| = \prod'_{p \in \mathcal{P}} |(1 - p^{-s})^{-1}|$.*

Proof. By theorems 386 and 387 with $P = \mathcal{P}$ and $w(p) = (1 - p^{-s})^{-1}$, which is multipiable. □

Lemma 389 (Abs zeta)**.** *For $s \in \mathbb{C}$ with $\Re(s) > 1$, we have $|\zeta(s)| = \prod'_{p \in \mathcal{P}} |(1 - p^{-s})^{-1}|$.*

Proof. By theorems 386 and 388. □

Lemma 390 (Abs inverse)**.** *For $z \in \mathbb{C}$, if $z \neq 0$ then $|z^{-1}| = |z|^{-1}$.*

Proof. Mathlib: abs_inv □

Lemma 391 (Nonzero factor)**.** *For $p \in \mathcal{P}$ and $s \in \mathbb{C}$ with $\Re(s) > 1$, we have $1 - p^{-s} \neq 0$.*

Proof. By theorem 385. □

Lemma 392 (Abs product)**.** *For $s \in \mathbb{C}$ with $\Re(s) > 1$, we have $|\zeta(s)| = \prod'_{p \in \mathcal{P}} |1 - p^{-s}|^{-1}$.*

Proof. Apply theorem 389 and theorem 390 with $z = 1 - p^{-s}$. Note $z \neq 0$ by theorem 391. □

Lemma 393 (Real double)**.** *For* $s \in \mathbb{C}$ *we have* $\Re(2s) = 2\Re(s)$*.*

Proof. □

Lemma 394 (Real bound)**.** *For* $s \in \mathbb{C}$*, if* $\Re(s) > 1$ *then* $\Re(2s) > 1$*.*

Proof. By theorem 393 and assumption $\Re(s) > 1$. □

Lemma 395 (Zeta ratio)**.** *For* $s \in \mathbb{C}$ *with* $\Re(s) > 1$*, we have* $\frac{\zeta(2s)}{\zeta(s)} = \frac{\prod'_{p\in\mathcal{P}}(1-p^{-2s})^{-1}}{\prod'_{p\in\mathcal{P}}(1-p^{-s})^{-1}}$*.*

Proof. Apply Lemma 386 twice, to both $\zeta(2s)$ and $\zeta(s)$. Use condition theorem 394. □

Lemma 396 (Ratio product)**.** *Let* P *be a set, and* $a(p) : P \to \mathbb{C}$ *and* $b(p) : P \to \mathbb{C}$ *be multipliable. Then* $\frac{\prod'_{p\in P} a(p)}{\prod'_{p\in P} b(p)} = \prod'_{p\in P} \frac{a(p)}{b(p)}$*.*

Proof. We proceed by cases on whether a ever takes the value zero.

Case 1: There exists $p_0 \in P$ **such that** $a(p_0) = 0$**.**

In this case, the infinite product $\prod'_{p\in P} a(p)$ contains the factor $a(p_0) = 0$, and therefore:

$$\prod_{p\in P}{}' a(p) = 0$$

Similarly, the quotient function $p \mapsto a(p)/b(p)$ satisfies $(a/b)(p_0) = a(p_0)/b(p_0) = 0/b(p_0) = 0$, so:

$$\prod_{p\in P}{}' \frac{a(p)}{b(p)} = 0$$

Therefore, both sides of the desired equality equal zero:

$$\frac{\prod'_{p\in P} a(p)}{\prod'_{p\in P} b(p)} = \frac{0}{\prod'_{p\in P} b(p)} = 0 = \prod_{p\in P}{}' \frac{a(p)}{b(p)}$$

Case 2: For all $p \in P$**,** $a(p) \neq 0$**.**

In this case, our hypotheses are that both a and b are multipliable and map to non-zero values everywhere. Our strategy is to apply Assumption `tprod_div`, but doing so requires careful reasoning about the algebraic structures involved.

The Obstacle Assumption `tprod_div` requires the functions to map into a **Commutative Group**. The field of complex numbers, $\mathbb{C}$, is not a commutative group under multiplication because the element 0 lacks a multiplicative inverse. Therefore, we cannot directly apply the theorem to our functions a and b.

The Strategy: Lifting to the Group of Units The solution is to work within the **group of units of** $\mathbb{C}$, denoted $\mathbb{C}^\times$, which is the set of non-zero complex numbers $\mathbb{C} \setminus \{0\}$. This set *is* a commutative group under multiplication. Since we are in the case where $a(p)$ and $b(p)$ are always non-zero, we can "lift" our functions to have $\mathbb{C}^\times$ as their codomain.

Defining the Lifted Functions We define two new unit-valued functions, u and v:

$$u : P \to \mathbb{C}^\times, \quad u(p) := a(p) \tag{3.1}$$

$$v : P \to \mathbb{C}^\times, \quad v(p) := b(p) \tag{3.2}$$

These functions are well-defined because our case assumption ($\forall p, a(p) \neq 0$) and the given hypothesis ($\forall p, b(p) \neq 0$) guarantee their outputs are always in $\mathbb{C}^\times$.

Multipliability of the Lifted Functions The multipliability of u and v follows directly from that of a and b. A function's multipliability depends on the summability of $|f(p) - 1|$ over its support. Since the values of $u(p)$ and $a(p)$ are identical (and likewise for v and b), their multipliability properties are preserved.

- $\{p : u(p) \neq 1\} = \{p : a(p) \neq 1\}$ is countable (since a is multipliable).
- $\sum_p |u(p) - 1| = \sum_p |a(p) - 1| < \infty$ (since a is multipliable).
- Similarly for v and b.

Applying the Division Theorem With u and v established as multipliable functions into the commutative group $\mathbb{C}^\times$, we can now safely apply Assumption `tprod_div`. This gives us an equality that holds within $\mathbb{C}^\times$:

$$\frac{\prod'_{p \in P} u(p)}{\prod'_{p \in P} v(p)} = \prod'_{p \in P} \frac{u(p)}{v(p)} \tag{3.3}$$

Returning to $\mathbb{C}$ Our final step is to show that this equality in $\mathbb{C}^\times$ implies the desired equality in $\mathbb{C}$. This is true because the natural inclusion (coercion) from $\mathbb{C}^\times$ to $\mathbb{C}$ preserves the algebraic operations of division and infinite products. Since $\mathrm{coe}(u(p)) = a(p)$ and $\mathrm{coe}(v(p)) = b(p)$, applying this coercion to both sides of Equation (3.3) directly yields our goal:

$$\frac{\prod'_{p \in P} a(p)}{\prod'_{p \in P} b(p)} = \prod'_{p \in P} \frac{a(p)}{b(p)}$$

In both cases, the desired equality holds. □

Lemma 397 (Ratio split)**.** *For $s \in \mathbb{C}$ with $\Re(s) > 1$, we have* $\frac{\prod'_{p \in \mathcal{P}} (1-p^{-2s})^{-1}}{\prod'_{p \in \mathcal{P}} (1-p^{-s})^{-1}} = \prod'_{p \in \mathcal{P}} \frac{(1-p^{-2s})^{-1}}{(1-p^{-s})^{-1}}$.

Proof. By theorem 396 with $a(p) = (1 - p^{-2s})^{-1}$ and $b(p) = (1 - p^{-s})^{-1}$. Multipliability holds by theorem 386, and use condition theorem 394. □

Lemma 398 (Ratio form)**.** *For $s \in \mathbb{C}$ with $\Re(s) > 1$, we have* $\frac{\zeta(2s)}{\zeta(s)} = \prod'_{p \in \mathcal{P}} \frac{(1-p^{-2s})^{-1}}{(1-p^{-s})^{-1}}$.

Proof. By theorems 395 and 397. □

Lemma 399 (Diff squares)**.** *For any $z \in \mathbb{C}$, we have $(1 - z^2) = (1 - z)(1 + z)$.*

Proof. Basic algebra □

Lemma 400 (Inverse ratio)**.** *For any $z \in \mathbb{C}$. If $|z| < 1$ then* $\frac{(1-z^2)^{-1}}{(1-z)^{-1}} = (1 + z)^{-1}$.

Proof. By theorem 399, then invert terms and simplify. Note $|z| < 1$ implies $z \neq \pm 1$ so we may invert $1 - z$ and $1 + z$ and $1 - z^2$. □

Theorem 401 (Ratio identity)**.** *For $s \in \mathbb{C}$ with $\Re(s) > 1$, we have* $\frac{\zeta(2s)}{\zeta(s)} = \prod'_{p \in \mathcal{P}} (1 + p^{-s})^{-1}$.

Proof. Apply Lemma 398 and Lemma 400 with $z = p^{-s}$. We verify condition using theorem 385. □

Lemma 402 (Three halves)**.** *We have* $\frac{\zeta(3)}{\zeta(3/2)} = \prod'_{p \in \mathcal{P}} (1 + p^{-3/2})^{-1}$.

Proof. Apply Theorem 401 with $s = 3/2$. Note $\Re(3/2) = 3/2 > 1$. $\square$

Lemma 403 (Triangle abs). *For any $z \in \mathbb{C}$, we have $|1 - z| \leq 1 + |z|$.*

Proof. By the triangle inequality, $|a + b| \leq |a| + |b|$. Let $a = 1$ and $b = -z$. Then $|1 - z| \leq |1| + |-z| = 1 + |z|$. $\square$

Lemma 404 (Prime power). *For $p \in \mathcal{P}$ and $s = \sigma + it \in \mathbb{C}$, we have $|p^{-s}| = p^{-\sigma}$.*

Proof. $|p^{-s}| = |p^{-\sigma - it}| = |p^{-\sigma} p^{-it}| = |p^{-\sigma}||e^{-it \log p}| = p^{-\sigma} \cdot 1 = p^{-\sigma}$. $\square$

Lemma 405 (Term bound). *For $p \in \mathcal{P}$ and $t \in \mathbb{R}$, we have $|1 - p^{-(3/2+it)}| \leq 1 + p^{-3/2}$.*

Proof. Apply Lemma 403 with $z = p^{-(3/2+it)}$. This gives $|1 - p^{-(3/2+it)}| \leq 1 + |p^{-(3/2+it)}|$. Apply Lemma 404 with $\sigma = 3/2$ to get $|p^{-(3/2+it)}| = p^{-3/2}$. $\square$

Lemma 406 (Inv order). *If $0 < a \leq b$, then $a^{-1} \geq b^{-1}$.*

Proof. Basic property of inequalities $\square$

Lemma 407 (Nonzero term). *For $p \in \mathcal{P}$, we have $1 - p^{-(3/2+it)} \neq 0$.*

Proof. We have $p^{-(3/2+it)} \neq 1$ by theorem 385 with $s = 3/2 + it$. $\square$

Lemma 408 (Inverse bound). *For $p \in \mathcal{P}$ and $t \in \mathbb{R}$, we have $|1 - p^{-(3/2+it)}|^{-1} \geq (1 + p^{-3/2})^{-1}$.*

Proof. Apply theorem 405, and then theorems 406 and 407 with $a = |1 - p^{-(3/2+it)}|$ and $b = 1 + p^{-3/2}$. $\square$

Lemma 409 (Prod order). *Let P be a set, and $a(p) : P \to \mathbb{C}$ and $b(p) : P \to \mathbb{C}$ be multipliable. If $0 < a(p) \leq b(p)$ for all $p \in P$ then $\prod'_{p \in P} a(p) \leq \prod'_{p \in P} b(p)$.*

Proof. Mathlib: tprod_le_tprod $\square$

Lemma 410 (Zeta compare). *For $t \in \mathbb{R}$, we have $\prod'_{p \in \mathcal{P}} (1 + p^{-3/2})^{-1} \leq \prod'_{p \in \mathcal{P}} |1 - p^{-(3/2+it)}|^{-1}$.*

Proof. Apply theorems 408 and 409 with $P = \mathcal{P}$, $a(p) = (1 + p^{-3/2})^{-1}$, $b(p) = |1 - p^{-(3/2+it)}|^{-1}$. Multipliability holds by theorem 386 $\square$

Theorem 411 (Zeta lower). *For any $t \in \mathbb{R}$, we have $|\zeta(3/2 + it)| \geq \frac{\zeta(3)}{\zeta(3/2)}$.*

Proof. From Lemma 392 with $s = 3/2 + it$, the left hand side is $|\zeta(3/2 + it)| = \prod'_{p \in \mathcal{P}} |1 - p^{-(3/2+it)}|^{-1}$. From Lemma 402, the right hand side is $\frac{\zeta(3)}{\zeta(3/2)} = \prod'_{p \in \mathcal{P}} (1 + p^{-3/2})^{-1}$. The theorem then follows directly from the inequality in Lemma 410. $\square$

Lemma 412 (Zeta positive). *For $x \in \mathbb{R}$, if $x > 1$ then $\zeta(x) \in \mathbb{R}$ and $\zeta(x) > 0$.*

Proof. By zeta_eq_tsum_one_div_nat_add_one_cpow, since $x > 1$ we have $\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$. Then note n^{-x} is positive real for all n, so the sum is also positive real. $\square$

Lemma 413 (Ratio positive). *We have $\frac{\zeta(3)}{\zeta(3/2)} > 0$.*

Proof. By theorem 412 applied twice, to both $x = 3$ and $x = 3/2$. $\square$

Lemma 414 (Fixed lower). *There exists $a > 0$ such that for any $t \in \mathbb{R}$, we have $|\zeta(3/2 + it)| \geq a$.*

Proof. By theorem 411 with $a = \frac{\zeta(3)}{\zeta(3/2)}$. Note $a > 0$ by theorem 413. $\square$

3.2 Zeta bound

Lemma 415 (Series form)**.** *For $s \in \mathbb{C}$ with $\Re(s) > 1$, we have $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.*

Proof. Mathlib: zeta_eq_tsum_one_div_nat_add_one_cpow ☐

Definition 416 (Partial sum)**.** For $s \in \mathbb{C}$ and $N \in \mathbb{N}$, define the partial sum $\zeta_N(s) = \sum_{n=1}^{N} n^{-s}$.

Lemma 417 (Abel sum)**.** *Let $a_n \in \mathbb{C}$ and let $f : \mathbb{R} \to \mathbb{C}$ be a continuously differentiable function. Let $A(u) = \sum_{n=1}^{\lfloor u \rfloor} a_n$. Then for any integer $N \geq 1$,*

$$\sum_{n=1}^{N} a_n f(n) = A(N)f(N) - \int_1^N A(u)f'(u)du.$$

Proof. Mathlib: sum_mul_eq_sub_sub_integral_mul ☐

Lemma 418 (Sum identity)**.** *For $s \in \mathbb{C}$, let $f(u) = u^{-s}$ and $a_n = 1$ for all n $N \in \mathbb{N}$. Then $\zeta_N(s) = \sum_{n=1}^{N} a_n f(n)$.*

Proof. Direct substution ☐

Lemma 419 (Count sum)**.** *Let $a_n = 1$. For $u \geq 1$, let $A(u) = \sum_{n=1}^{\lfloor u \rfloor} a_n$. Then $A(u) = \lfloor u \rfloor$.*

Proof. By definition, $\sum_{n=1}^{\lfloor u \rfloor} 1 = \lfloor u \rfloor$. ☐

Lemma 420 (Power deriv)**.** *Let $f(u) = u^{-s}$. Then $f'(u) = -su^{-s-1}$.*

Proof. Apply the power rule for differentiation. See Mathlib/Analysis/Calculus. ☐

Lemma 421 (Apply Abel)**.** *For $s \in \mathbb{C}$ and integer $N \geq 1$,*

$$\zeta_N(s) = \lfloor N \rfloor N^{-s} - \int_1^N \lfloor u \rfloor(-su^{-s-1})du.$$

Proof. Apply Lemma 417 with $f(u) = u^{-s}$ and $a_n = 1$. Use lemma 418, and $A(u)$ from Lemma 419, and $f'(u)$ from Lemma 420. ☐

Lemma 422 (Floor int)**.** *For an integer $N \geq 1$, $\lfloor N \rfloor = N$.*

Proof. By definition of the floor function. ☐

Lemma 423 (First form)**.** *For $s \in \mathbb{C}$ and integer $N \geq 1$,*

$$\zeta_N(s) = N^{1-s} + s \int_1^N \lfloor u \rfloor u^{-s-1} du.$$

Proof. Apply Lemma 421 and Lemma 422. ☐

Lemma 424 (Floor split)**.** *For any $u \in \mathbb{R}$, $\lfloor u \rfloor = u - \{u\}$, where $\{u\}$ is the fractional part of u.*

Proof. By definition of the fractional part function. ☐

Lemma 425 (Integral split)**.** *For $s \in \mathbb{C}$ and integer $N \geq 1$,*

$$\int_1^N \lfloor u \rfloor u^{-s-1} du = \int_1^N u^{-s} du - \int_1^N \{u\} u^{-s-1} du.$$

Proof. Apply Lemma 424 and linearity of the integral. □

Lemma 426 (Second form)**.** *For $s \in \mathbb{C}$ and integer $N \geq 1$,*

$$\zeta_N(s) = N^{1-s} + s \int_1^N u^{-s} du - s \int_1^N \{u\} u^{-s-1} du.$$

Proof. Apply Lemmas 423 and 425. □

Lemma 427 (Main integral)**.** *For $s \in \mathbb{C}, s \neq 1$, we have $s \int_1^N u^{-s} du = \frac{s}{1-s}(N^{1-s} - 1)$.*

Proof. The antiderivative of u^{-s} is $\frac{u^{1-s}}{1-s}$. Evaluate at $u = N$ and $u = 1$. □

Lemma 428 (Final form)**.** *For $s \in \mathbb{C}, s \neq 1$ and integer $N \geq 1$,*

$$\zeta_N(s) = \frac{N^{1-s}}{1-s} + 1 + \frac{1}{s-1} - s \int_1^N \{u\} u^{-s-1} du.$$

Proof. Apply Lemmas 426 and 427 and combine terms: $N^{1-s} + \frac{s}{1-s} N^{1-s} = N^{1-s}(1 + \frac{s}{1-s}) = N^{1-s}(\frac{1-s+s}{1-s}) = \frac{N^{1-s}}{1-s}$. The term $-\frac{s}{1-s}$ is $\frac{s}{s-1} = 1 + \frac{1}{s-1}$. □

Lemma 429 (Limit term)**.** *If $\Re(s) > 1$, then $\lim_{N\to\infty} N^{1-s} = 0$.*

Proof. $|N^{1-s}| = N^{1-\Re(s)}$. Since $1 - \Re(s) < 0$, this limit tends to 0. □

Lemma 430 (Frac bound)**.** *For any $u \in \mathbb{R}$, $0 \leq \{u\} < 1$, and thus $|\{u\}| \leq 1$.*

Proof. By definition of the fractional part. □

Lemma 431 (Term bound)**.** *For $u \geq 1$ and $s \in \mathbb{C}$, $|\{u\} u^{-s-1}| \leq u^{-\Re(s)-1}$.*

Proof. Apply Lemma 430. We have $|\{u\} u^{-s-1}| = |\{u\}||u^{-s-1}| \leq 1 \cdot u^{-\Re(s)-1}$. □

Lemma 432 (Eps bound)**.** *Let $\varepsilon > 0$ and $u \geq 1$. If $\Re(s) \geq \varepsilon$ then $|\{u\} u^{-s-1}| \leq u^{-1-\varepsilon}$.*

Proof. Apply Lemma 431 and that $x \mapsto u^{-1-x}$ is monotonic. □

Lemma 433 (Triangle int)**.** *For $z_1, z_2 \in \mathbb{C}$, $|z_1 + z_2| \leq |z_1| + |z_2|$. For an integral, $|\int g(u) du| \leq \int |g(u)| du$.*

Proof. Standard results from complex analysis. □

Lemma 434 (Integral conv)**.** *Let $\varepsilon > 0$ If $\Re(s) \geq \varepsilon$, the integral $\int_1^\infty \{u\} u^{-s-1} du$ converges uniformly.*

Proof. By Lemmas 433 and 432, we calculate

$$\Big| \int_1^\infty \{u\} u^{-s-1} du \Big| \leq \int_1^\infty |\{u\} u^{-s-1}| \leq \int_1^\infty u^{-1-\varepsilon} du = \frac{1}{\varepsilon}.$$

Thus the integral converges. □

Lemma 435 (Zeta formula)**.** *For* $\Re(s) > 1$*,*

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \{u\}u^{-s-1}du.$$

Proof. Take the limit $N \to \infty$ in Lemma 428. Apply Lemmas 415, 429, and 434. ☐

Lemma 436 (Analytic off)**.** *Let* $S = \{s \in \mathbb{C} : s \neq 1\}$*. Then* $\zeta(s)$ *is analyticOnNhd* S*.*

Proof. Apply theorem 464 ☐

Lemma 437 (S open)**.** *Let* $S = \{s \in \mathbb{C} : s \neq 1\}$*. Then* S *is open.*

Proof. S is the complement of the singleton $\{1\}$, which is open. ☐

Lemma 438 (T open)**.** *Let* $S = \{s \in \mathbb{C} : s \neq 1\}$ *and* $T = \{s \in S : \Re(s) > 1/10\}$*. Then* T *is open.*

Proof. T is the intersection of the open set S with the open half-plane $\{s : \Re(s) > 1/10\}$. ☐

Lemma 439 (T connected)**.** *Let* $S = \{s \in \mathbb{C} : s \neq 1\}$ *and* $T = \{s \in S : \Re(s) > 1/10\}$*. Then* T *is preconnected.*

Proof. The set T can be shown to be path-connected, which implies preconnected. ☐

Lemma 440 (Integral analytic)**.** *If the integral of an analytic function* $f : \mathbb{C} \to \mathbb{C}$ *converges uniformly for all* s *such that* $\Re(s) \geq \frac{1}{10}$*, then the integral is analytic (as a function of* s*).*

Proof. ☐

Lemma 441 (Analytic ext)**.** *Let* $S = \{s \in \mathbb{C} : s \neq 1\}$ *and* $T = \{s \in S : \Re(s) > 1/10\}$*. The function* $F(z) = \frac{z}{z-1} - z\int_1^\infty \{u\}u^{-z-1}du$ *is analyticOnNhd* T*.*

Proof. Take $s \in T$. The function $\frac{z}{z-1}$ is analyticAt $z = s$, since $s \neq 1$. The integral converges uniformly by theorem 434, so $F(z)$ is analyticAt $z = s$. ☐

Lemma 442 (Divide split)**.** *For any complex number* $z \neq 1$*, we have* $\frac{z}{z-1} = 1 + \frac{1}{z-1}$*.*

Proof. Direct algebraic manipulation: $\frac{z}{z-1} = \frac{(z-1)+1}{z-1} = \frac{z-1}{z-1} + \frac{1}{z-1} = 1 + \frac{1}{z-1}$. ☐

Lemma 443 (Zeta extend)**.** *Let* $S = \{s \in \mathbb{C} : s \neq 1\}$ *and* $T = \{s \in S : \Re(s) > 1/10\}$*. We have* $\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \{u\}u^{-s-1}du$ *on* T*.*

Proof. By Lemma 435, the equality $\zeta(s) = F(s)$ holds for $\Re(s) > 1$. By Lemma 441, $F(s)$ is analyticOnNhd T. By Lemma 436 $\zeta(s)$ is analyticOnNhd $S \supset T$. Hence by the identity principle, (Mathlib try AnalyticOnNhd.eqOn_of_preconnected_of_eventuallyEq) the equality $\zeta(s) = F(s)$ holds in T. ☐

Lemma 444 (First bound)**.** *For* $\Re(s) > 0, s \neq 1$*,*

$$|\zeta(s)| \leq 1 + \left|\frac{1}{s-1}\right| + |s|\int_1^\infty |\{u\}u^{-s-1}|du.$$

Proof. Apply Lemma 433 to the formula in Lemma 443. ☐

Lemma 445 (Integral value)**.** *For* $\Re(s) > 0$*,* $\int_1^\infty u^{-\Re(s)-1}du = \frac{1}{\Re(s)}$*.*

Proof. The antiderivative is $\frac{u^{-\Re(s)}}{-\Re(s)}$. Evaluating from 1 to ∞ gives $0 - \frac{1}{-\Re(s)} = \frac{1}{\Re(s)}$. □

Lemma 446 (Second bound)**.** *For $\Re(s) > 0, s \neq 1$,*

$$|\zeta(s)| \leq 1 + \left|\frac{1}{s-1}\right| + \frac{|s|}{\Re(s)}.$$

Proof. Apply Lemmas 444, 431, and 445. □

Lemma 447 (Inverse mod)**.** *For $s \in \mathbb{C}, s \neq 1$, we have $\left|\frac{1}{s-1}\right| = \frac{1}{|s-1|}$.*

Proof. Algebraic identity and Lemma 433. □

Lemma 448 (Third bound)**.** *For $\Re(s) > 0, s \neq 1$,*

$$|\zeta(s)| \leq 1 + \frac{1}{|s-1|} + \frac{|s|}{\Re(s)}.$$

Proof. Apply Lemmas 446 and 447. □

Lemma 449 (s bound)**.** *Let $s = \sigma + it$. If $\frac{1}{2} \leq \sigma < 3$, then $|s| < 3 + |t|$.*

Proof. $|s|^2 = \sigma^2 + t^2 \leq 3^2 + t^2 = 9 + |t|^2$. Since $0 \leq 6|t|$, we have $9 + |t|^2 \leq 9 + 6|t| + |t|^2 = (3 + |t|)^2$. Taking the square root gives $|s| \leq 3 + |t|$. □

Lemma 450 (Real inv)**.** *If $\frac{1}{2} \leq \Re(s) < 3$, then $\dfrac{1}{\Re(s)} \leq 2$.*

Proof. From $1/2 \leq \Re(s)$, taking reciprocals reverses the inequality. □

Lemma 451 (Shift bound)**.** *Let $s = \sigma + it$. If $\frac{1}{2} \leq \sigma < 3$ and $|t| \geq 1$, then $|s - 1| \geq 1$.*

Proof. $|s-1|^2 = (\sigma - 1)^2 + t^2$. Since $|t| \geq 1$, $t^2 \geq 1$. Since $(\sigma - 1)^2 \geq 0$, we have $|s-1|^2 \geq 1$. □

Lemma 452 (Combine bounds)**.** *If $s = \sigma + it$ with $\frac{1}{2} \leq \sigma < 3$ and $|t| \geq 1$, then*

$$|\zeta(s)| < 1 + 1 + (3 + |t|) \cdot 2.$$

Proof. In Lemma 448, apply Lemma 451 to bound $\frac{1}{|s-1|} \leq 1$. Apply Lemma 449 to bound $|s|$ and Lemma 450 to bound $\frac{1}{\Re(s)}$. □

Lemma 453 (Algebra step)**.** *For $|t| \geq 1$, we have $1 + 1 + (3 + |t|) \cdot 2 = 8 + 2|t|$.*

Proof. By arithmetic. $2 + 6 + 2|t| = 8 + 2|t|$. □

Lemma 454 (Upper bound)**.** *For all $z \in \mathbb{C}$ with $\frac{1}{2} \leq \Re(z) < 3$ and $|\Im(z)| \geq 1$, we have $|\zeta(z)| < 8 + 2|\Im(z)|$.*

Proof. Apply Lemmas 452 and 453. □

Lemma 455 (Shift calc)**.** *For $s \in \mathbb{C}$, $t \in \mathbb{R}$ let $z = s + 3/2 + it$. Then $\Re(z) = \Re(s) + 3/2$ and $\Im(z) = \Im(s) + t$.*

Proof. Direct calculation □

Lemma 456 (Shift cond)**.** *For $s \in \mathbb{C}$, $t \in \mathbb{R}$ let $z = s + 3/2 + it$. If $|s| \leq 1$ and $|t| \geq 3$, then $\Re(z) \in [1/2, 3]$ and $|\Im(z)| \geq 1$*

Proof. Apply theorem 455, and use arithmetic. Here $\Im(s)^2 + \Re(s)^2 = |s|^2 \in [0,1]$ by assumption. □

Lemma 457 (Global bound). *There exists $b > 1$ such that for all $t \in \mathbb{R}$ we have $|\zeta(s+3/2+it)| \leq 8 + 2|t|$ for all $|s| \leq 1$ and $|t| \geq 3$.*

Proof. Apply theorems 454 and 456. □

3.3 Zeta derivatives

Lemma 458 (Diff off pole). *Let $S = \{s \in \mathbb{C} : s \neq 1\}$. For all $s \in S$ we have $\zeta(s)$ DifferentiableAt s.*

Proof. □

Lemma 459 (At to within). *Let $T \subset \mathbb{C}$. For $g : T \to \mathbb{C}$ and $s \in T$, if g DifferentiableAt s then g DifferentiableWithinAt s*

Proof. Mathlib: DifferentiableAt.differentiableWithinAt □

Lemma 460 (Within to on). *Let $T \subset \mathbb{C}$. For $g : T \to \mathbb{C}$, if g DifferentiableWithinAt s for all $s \in T$, then g DifferentiableOn T*

Proof. Unfold definition of DifferentiableOn T in terms of differentiableWithinAt s for all $s \in T$. □

Lemma 461 (At to on). *Let $T \subset \mathbb{C}$. For $g : T \to \mathbb{C}$, if g DifferentiableAt s for all $s \in T$, then g DifferentiableOn T*

Proof. By theorems 459 and 460 □

Lemma 462 (Diff to anal). *Let open $T \subset \mathbb{C}$. For $g : T \to \mathbb{C}$, if g DifferentiableOn T, then g analyticOnNhd T*

Proof. Mathlib: Complex.analyticOnNhd_iff_differentiableOn □

Lemma 463 (At gives anal). *Let open $T \subset \mathbb{C}$. For $g : T \to \mathbb{C}$, if g DifferentiableAt s for all $s \in T$, then g analyticOnNhd T.*

Proof. By theorems 461 and 462 □

Lemma 464 (Analytic off). *Let $S = \{s \in \mathbb{C} : s \neq 1\}$. Then $\zeta(s)$ is analyticOnNhd S.*

Proof. Apply theorems 458 and 463 with $T = S$ and $g(s) = \zeta(s)$. □

Lemma 465 (Disk avoid). *Let $t \in \mathbb{R}$ with $|t| > 1$. Let $c = 3/2 + it$ and $S_t = \{s \in \mathbb{C} : s + c \neq 1\}$. Then $s \neq 1$ for all $s \in \mathbb{C}$ with $|s - c| \leq 1$.*

Proof. For sake of contradiction, suppose $s = 1$. Then we calculate

$$|s - c| = |1 - c| = |1 - 3/2 - it| = |1/2 - it| \geq |\Im(it)| = |t|.$$

Thus $|s - c| \geq |t| > 1$, but this contradicts $|s - c| \leq 1$. Hence the proof is complete. □

Lemma 466 (Disk subset). *Let $t \in \mathbb{R}$ with $|t| > 1$. Let $c = 3/2 + it$. Then $\overline{\mathbb{D}}_1(c) \subset S$*

Proof. By theorem 465, and unfolding the definitions of c, $\overline{\mathbb{D}}_1(c)$, and S. □

Lemma 467 (Disk analytic)**.** *Let $t \in \mathbb{R}$ with $|t| > 1$, $x \in \mathbb{R}$, and let $c = x + it$. Then $\zeta(s)$ is analyticOnNhd $\overline{\mathbb{D}}_1(c)$.*

Proof. Apply theorems 464 and 466, and then Mathlib: AnalyticOnNhd.mono □

Lemma 468 (Zero free)**.** *Let $s \in \mathbb{C}$. If $\Re(s) > 1$ then $\zeta(s) \neq 0$.*

Proof. □

Lemma 469 (Point nonzero)**.** *For all $t \in \mathbb{R}$ we have $\zeta(3/2 + it) \neq 0$*

Proof. By theorem 468 □

Lemma 470 (Normalize analytic)**.** *Let $c \in \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1(c)$ with $f(c) \neq 0$. Then the function $f_c(z) = f(z + c)/f(c)$ is AnalyticOnNhd $\overline{\mathbb{D}}_1$ and satisfies $f_c(0) = 1$.*

Proof. Since f is AnalyticOnNhd $\overline{\mathbb{D}}_1(c)$, and $\overline{\mathbb{D}}_1(c) = \{z+c : z \in \overline{\mathbb{D}}_1\}$, then f_c is AnalyticOnNhd $\overline{\mathbb{D}}_1$.

Next we calculate $f_c(0) = \frac{f(0+c)}{f(c)} = \frac{f(c)}{f(c)} = 1$. □

Lemma 471 (Log derivative)**.** *Let $c \in \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1(c)$ with $f(c) \neq 0$. Let $f_c(z) = f(z + c)/f(c)$. Then for any z where $f(z + c) \neq 0$, we have logDeriv$(f_c)(z)$ = logDeriv$(f)(z + c)$.*

Proof. By the chain rule, we calculate $f_c'(z) = f'(z + c)$. Thus since $f(c), f(z + c) \neq 0$, we calculate

$$\text{logDeriv}(f_c)(z) = \frac{f_c'(z)}{f_c(z)} = \frac{f'(z + c)/f(c)}{f(z + c)/f(c)} = \frac{f'(z + c)}{f(z + c)} = \text{logDeriv}(f)(z + c)$$

□

Lemma 472 (Shift bound)**.** *Let $B > 1$, $0 < R < 1$, $c \in \mathbb{C}$, and $f : \mathbb{C} \to \mathbb{C}$ with $f(c) \neq 0$. If $|f(z)| \leq B$ for all $z \in \overline{\mathbb{D}}_R(c)$, then the function $f_c(z) = f(z + c)/f(c)$ satisfies $|f_c(z)| \leq B/|f(c)|$ for all $z\overline{\mathbb{D}}_R$.*

Proof. If $z\overline{\mathbb{D}}_R$ then $z + c \in \overline{\mathbb{D}}_R$, so $|f(z + c)| \leq B$ by assumption. Thus we calculate $|f_c(z)| = |f(z + c)|/f(c) \leq B/|f(c)|$. □

Lemma 473 (Zero shift)**.** *Let $r > 0$, $c \in \mathbb{C}$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1(c)$ with $f(c) \neq 0$. Let $f_c(z) = f(z + c)/f(c)$. We have $\rho' \in \mathcal{K}_{f_c}(r)$ if and only if $\rho' = \rho - c$ where $\rho \in \mathcal{K}_f(r; c)$. In particular $\mathcal{K}_{f_c}(r) = \{\rho - c : \rho \in \mathcal{K}_f(r)\}$.*

Proof. By definition, $\rho' \in \mathcal{K}_{f_c}(r)$ means $f_c(\rho') = 0$ and $|\rho'| \leq r$. By definition of f_c we have $f_c(\rho') = f(\rho' + c)/f(c)$. Since $f(c) \neq 0$ we conclude $f(\rho' + c) = 0$. Also $|(\rho' + c) - c| = |\rho'| \leq r$, and hence $\rho' + c \in \mathcal{K}_f(r; c)$. Therefore $\rho' \in \mathcal{K}_{f_c}(r)$ implies $\rho' + c \in \mathcal{K}_f(r; c)$

The proof that $\rho \in \mathcal{K}_f(r; c)$ implies $\rho - c \in \mathcal{K}_{f_c}(r)$ is similar. □

Lemma 474 (Order shift)**.** *Let $r > 0$, $c \in \mathbb{C}$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1(c)$ with $f(c) \neq 0$. Let $f_c(z) = f(z + c)/f(c)$. For $\rho \in \mathcal{K}_{f_c}(r)$, the analyticOrderAt satisfies $m_{\rho,f_c} = m_{\rho+c,f}$.*

Proof. By definition of analyticOrderAt, we have $f_c(z) = (z-\rho)^{m_{\rho,f_c}} h(z)$ for some h AnalyticAt ρ with $h(\rho) \neq 0$. As $f_c(z) = f(z+c)/f(c)$ and $f(c) \neq 0$, this implies $f(z+c) = (z-\rho')^{m_{\rho',f_c}} h(z) f(c)$. Thus letting $w = z+c$ and $g(w) = h(w-c)f(c)$, we have $f(w) = (w-c-\rho)^{m_{\rho,f_c}} g(w)$. Observe h AnalyticAt ρ' implies that g is AnalyticAt $\rho'+c$. And $h(\rho'), f(c) \neq 0$ imply $g(\rho+c) \neq 0$. Hence by definition we conclude AnalyticAt of f at $\rho + c$ equals m_{ρ',f_c}. □

Lemma 475 (Disk minus K)**.** *Let $r_1 > 0$, $c \in \mathbb{C}$, and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1(c)$ with $f(c) \neq 0$. Let $f_c(z) = f(z+c)/f(c)$. We have $z \in \overline{\mathbb{D}}_{r_1} \setminus \mathcal{K}_{f_c}(R_1)$ if and only if $z + c \in \overline{\mathbb{D}}_{r_1}(c) \setminus \mathcal{K}_f(R_1; c)$*

Proof. First $z \in \overline{\mathbb{D}}_{r_1}$ if and only if $|z| \leq r_1$ if and only if $|(z+c)-c| \leq r_1$ iff $z + c \in \overline{\mathbb{D}}_{r_1}(c)$. Second, since $f(c) \neq 0$ we have $z \in \mathcal{K}_{f_c}(R_1)$ if and only if $f_c(z) = 0$ if and only if $f(z+c) = 0$ if and only if $z + c \in \mathcal{K}_f(R_1; c)$. Combining these two equivalences, $z \in \overline{\mathbb{D}}_{r_1} \setminus \mathcal{K}_{f_c}(R_1)$ if and only if $z + c \in \overline{\mathbb{D}}_{r_1}(c) \setminus \mathcal{K}_f(R_1; c)$. □

Lemma 476 (Final bound)**.** *Let $B > 1$, $0 < r_1 < r < R_1 < R < 1$. Let $c \in \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ AnalyticOnNhd $\overline{\mathbb{D}}_1(c)$ with $f(c) \neq 0$. Let $f_c(z) = f(z+c)/f(c)$. If $|f(z)| < B$ for all $z \in \overline{\mathbb{D}}_R(c)$, then for all $z \in \overline{\mathbb{D}}_{r_1} \setminus \mathcal{K}_{f_c}(R_1)$ we have*

$$\left| \frac{f_c'}{f_c}(z) - \sum_{\rho' \in \mathcal{K}_{f_c}(R_1)} \frac{m_{\rho',f_c}}{z - \rho'} \right| \leq \left(\frac{16r^2}{(r-r_1)^3} + \frac{1}{(R^2/R_1 - R_1)\log(R/R_1)} \right) \log(B/|f(c)|).$$

Proof. Apply theorem 383 with the function f_c, using the conditions theorems 470 to 472. □

Lemma 477 (Log expansion)**.** *Let $t \in \mathbb{R}$ with $|t| > 3$. Let $c = 3/2 + it$, $B > 1$, $0 < r_1 < r < R_1 < R < 1$. If $|\zeta(z)| < B$ for all $z \in \overline{\mathbb{D}}_R(c)$, then for all $z \in \overline{\mathbb{D}}_{r_1}(c) \setminus \mathcal{K}_\zeta(R_1; c)$ we have*

$$\left| \frac{\zeta'(z)}{\zeta(z)} - \sum_{\rho \in \mathcal{K}_\zeta(R_1;c)} \frac{m_{\rho,\zeta}}{z - \rho} \right| \leq \left(\frac{16r^2}{(r-r_1)^3} + \frac{1}{(R^2/R_1 - R_1)\log(R/R_1)} \right) \log(B/|\zeta(c)|)$$

Proof. We apply theorem 476 using $f(z) = \zeta(z)$. The conditions ζ AnalyticOnNhd $\overline{\mathbb{D}}_1(c)$ with $\zeta(c) \neq 0$ hold by theorems 467 and 469. □

Lemma 478 (Lower shift)**.** *There exists $a > 0$ such that for all $t \in \mathbb{R}$ we have $|\zeta(3/2 + it)| \geq a$*

Proof. Euler product for zeta, triangle inequality, properties of $\zeta(\sigma)$ for $\sigma > 1$ Let $s = \sigma + it$. We are interested in the case where $\sigma = 3/2$.

Step 1: Use the Euler Product Formula

For any complex number s with $\Re(s) = \sigma > 1$, the Riemann zeta function can be represented by the absolutely convergent Euler product over all prime numbers p:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

This implies that its reciprocal is

$$\frac{1}{\zeta(s)} = \prod_p (1 - p^{-s})$$

We take the modulus of both sides:

$$\left|\frac{1}{\zeta(s)}\right| = \left|\prod_p (1-p^{-s})\right| = \prod_p |1-p^{-s}|$$

Step 2: Bound the term $|1-p^{-s}|$

Using the triangle inequality ($|z_1+z_2| \le |z_1|+|z_2|$), we can bound each term in the product:

$$|1-p^{-s}| \le |1| + |-p^{-s}| = 1 + |p^{-s}|$$

The modulus of p^{-s} is:

$$|p^{-s}| = |p^{-(\sigma+it)}| = |p^{-\sigma}p^{-it}| = |p^{-\sigma}||e^{-it\log p}| = p^{-\sigma}\cdot 1 = p^{-\sigma}$$

So, we have $|1-p^{-s}| \le 1+p^{-\sigma}$.

Step 3: Bound the entire product

Substituting this back into the product for the reciprocal's modulus:

$$\left|\frac{1}{\zeta(s)}\right| \le \prod_p (1+p^{-\sigma})$$

The product $\prod_p (1+p^{-\sigma})$ can be expanded:

$$(1+2^{-\sigma})(1+3^{-\sigma})(1+5^{-\sigma})\cdots = 1+2^{-\sigma}+3^{-\sigma}+5^{-\sigma}+6^{-\sigma}+\ldots$$

This expanded sum contains terms $n^{-\sigma}$ for all square-free integers n. This sum is strictly less than the sum over all integers $n \ge 1$:

$$\prod_p (1+p^{-\sigma}) < \sum_{n=1}^{\infty} \frac{1}{n^\sigma}$$

The sum on the right is, by definition, the Riemann zeta function evaluated at the real number σ, i.e., $\zeta(\sigma)$. Thus, we have established that for $\sigma > 1$:

$$\left|\frac{1}{\zeta(\sigma+it)}\right| < \zeta(\sigma)$$

Step 4: Conclude the proof

Taking the reciprocal of the inequality (and flipping the inequality sign) gives:

$$|\zeta(\sigma+it)| > \frac{1}{\zeta(\sigma)}$$

We are interested in the specific case $\sigma = 3/2$. For this value, we have:

$$|\zeta(3/2+it)| > \frac{1}{\zeta(3/2)}$$

The value $\zeta(3/2) = \sum_{n=1}^{\infty} n^{-3/2}$ is a convergent sum of positive terms, so it is a finite positive constant (approximately 2.612). We can therefore define our constant a to be $a = 1/\zeta(3/2)$. Since $\zeta(3/2) > 0$, we have $a > 0$. The inequality $|\zeta(3/2+it)| \ge a$ holds for all $t \in \mathbb{R}$. $\square$

Lemma 479 (Log lower). *There exists $A > 1$ such that for all $t \in \mathbb{R}$,*

$$\log\left(\frac{1}{|\zeta(3/2+it)|}\right) \le A$$

Proof. Let $a > 0$ be as in theorem 478, so $|\zeta(3/2+it)| \ge a$ for all $t \in \mathbb{R}$. Set $A = \max\{2, \log(1/a)\}$. Clearly $A > 1$ since $a > 0$ and $\log(1/a) > 0$. For any $t \in \mathbb{R}$, set $x = |\zeta(3/2+it)|$. Then $a \le x$, so $1/x \le 1/a$ and $\log(1/x) \le \log(1/a) \le A$. Also, $\log(1/x) < 2 \le A$ for $x > 1/2$. Thus $\log(1/x) \le A$ for all t. ☐

Lemma 480 (Upper pre). *There exists $b > 1$ such that for all $t \in \mathbb{R}$ we have $|\zeta(s+3/2+it)| \le b|t|$ for all $|s| \le 1$ and $|t| \ge 3$.*

Proof. Apply theorem 457 ☐

Lemma 481 (Upper disk). *There exists $b > 1$ such that for all $t \in \mathbb{R}$ with $|t| > 3$, letting $c = 3/2 + it$, we have $|\zeta(s)| \le b|t|$ for all $s \in \overline{\mathbb{D}}_1(c)$.*

Proof. The proof of this lemma is a direct application of theorem 480 by a change of variables.

Step 1: Recall the prerequisite lemma

Theorem 480 states that there exists a constant $b > 1$ such that for all $t \in \mathbb{R}$ with $|t| \ge 3$ and for all complex numbers $s_{pre} \in \mathbb{C}$ with $|s_{pre}| \le 1$, the following inequality holds:

$$|\zeta(s_{pre} + 3/2 + it)| \le b|t|$$

We will show that the conditions and conclusion of the current lemma perfectly align with this statement.

Step 2: Unpack the conditions of the current lemma

We are given the following conditions:

1. A real number t with $|t| > 3$.
2. A complex number $c = 3/2 + it$.
3. A complex number s which belongs to the closed disk of radius 1 centered at c, denoted $\overline{\mathbb{D}}_1(c)$.

The condition $s \in \overline{\mathbb{D}}_1(c)$ means, by definition, that the distance between s and c is at most 1:

$$|s - c| \le 1$$

Step 3: Define a new variable to match the prerequisite

Our goal is to bound $|\zeta(s)|$. Let's define a new variable, which we will call s_{pre}, in a way that relates our s to the argument of the zeta function in theorem 480. Let's set the argument of ζ in our lemma, which is s, equal to the argument of ζ in the prerequisite lemma, which is $s_{pre} + 3/2 + it$:

$$s = s_{pre} + 3/2 + it$$

Now, let's solve for s_{pre}:

$$s_{pre} = s - (3/2 + it)$$

Recognizing the definition $c = 3/2 + it$, this simplifies to:

$$s_{pre} = s - c$$

Step 4: Verify the conditions on the new variable

Theorem 480 requires that the variable s_{pre} satisfies $|s_{pre}| \leq 1$. Let's check if our definition of s_{pre} meets this condition. From Step 2, we know that for any $s \in \overline{\mathbb{D}}_1(c)$, we have $|s-c| \leq 1$. Substituting our definition from Step 3, this is exactly the condition:

$$|s_{pre}| \leq 1$$

Therefore, for any s that satisfies the conditions of our lemma, we can define $s_{pre} = s - c$, and this s_{pre} will satisfy the conditions of theorem 480.

Step 5: Apply the prerequisite lemma and conclude

We have established the following:

- We are given t with $|t| > 3$. This matches the condition on t in theorem 480.
- For any $s \in \overline{\mathbb{D}}_1(c)$, we can write $s = s_{pre} + c = s_{pre} + 3/2 + it$, where $s_{pre} = s - c$ satisfies $|s_{pre}| \leq 1$.

We can now apply the inequality from theorem 480 to the number $\zeta(s_{pre} + 3/2 + it)$. The lemma guarantees the existence of a constant $b > 1$ such that:

$$|\zeta(s_{pre} + 3/2 + it)| \leq b|t|$$

Since $s = s_{pre} + 3/2 + it$, this inequality is identical to:

$$|\zeta(s)| \leq b|t|$$

This holds for any $s \in \overline{\mathbb{D}}_1(c)$ and any $t \in \mathbb{R}$ with $|t| > 3$. This is precisely the statement we needed to prove. □

Lemma 482 (Expand bound)**.** *There exists a constant $A > 1$ such that for all $t \in \mathbb{R}$ with $|t| > 3$, $c = 3/2 + it$, $B > 1$, $0 < r_1 < r < R_1 < R < 1$, $z \in \overline{\mathbb{D}}_{r_1}(c) \setminus \mathcal{K}_\zeta(R_1; c)$ we have*

$$\left| \frac{\zeta'(z)}{\zeta(z)} - \sum_{\rho \in \mathcal{K}_\zeta(R_1;c)} \frac{m_{\rho,\zeta}}{z-\rho} \right| \leq \left(\frac{16r^2}{(r-r_1)^3} + \frac{1}{(R^2/R_1 - R_1)\log(R/R_1)} \right) \Big(\log|t| + \log(b) + A \Big)$$

Proof. We apply theorems 477, 479 and 481 with $B = bt$, and $C_1 = C/R$. □

Lemma 483 (Final expansion)**.** *Let $0 < r_1 < r < 5/6$. There exists constants $C > 1$ such that for all $t \in \mathbb{R}$ with $|t| > 3$, $c = 3/2 + it$, and $z \in \overline{\mathbb{D}}_{r_1}(c) \setminus \mathcal{K}_\zeta(5/6; c)$ we have*

$$\left| \frac{\zeta'(z)}{\zeta(z)} - \sum_{\rho \in \mathcal{K}_\zeta(5/6;c)} \frac{m_{\rho,\zeta}}{z-\rho} \right| \leq C \left(\frac{1}{(r-r_1)^3} + 1 \right) \log|t|$$

Proof. We apply theorem 482 and choose $R_1 = 5/6$, $R = 8/9$. Set $C = \left(16 + \frac{1}{(R^2/R_1 - R_1)\log(R/R_1)}\right)(1 + \log(b) + A)$. □

Chapter 4

Zero Free Region

Definition 484 (Log derivative)**.** For $s \in \mathbb{C}$ define $Z(s) = \frac{\zeta'(s)}{\zeta(s)}$.

Definition 485 (Zero set)**.** Define the set $\mathcal{Z} = \{\sigma + it \in \mathbb{C} : \sigma, t \in \mathbb{R} \text{ and } \zeta(\sigma + it) = 0\}$.

Definition 486 (Window zeros)**.** For $t \in \mathbb{R}$ define the set

$$\mathcal{Z}_t = \{\rho_1 = \sigma_1 + it_1 \in \mathbb{C} : \zeta(\rho_1) = 0 \text{ and } |\rho_1 - (3/2 + it)| \leq 5/6\}$$

Lemma 487 (Finite set)**.** *For each $t \in \mathbb{R}$ the set $\mathcal{Z}_t$ is finite.*

Proof. □

Lemma 488 (Reciprocal real)**.** *Let $z \in \mathbb{C}$. If $\Re(z) > 0$ then $\Re(1/z) > 0$.*

Proof. □

Lemma 489 (Zero free)**.** *Let $\sigma, t \in \mathbb{R}$. If $\sigma > 1$ then $\zeta(\sigma + it) \neq 0$.*

Proof. Use `lemma _root_.riemannZeta_ne_zero_of_one_le_re` in `Nonvanishing.lean` in Mathlib / NumberTheory / LSeries . □

Lemma 490 (Zero bound)**.** *Let $\sigma_1, t_1 \in \mathbb{R}$. If $\zeta(\sigma_1 + it_1) = 0$ then $\sigma_1 \leq 1$.*

Proof. Contrapositive of Lemma 489. □

Lemma 491 (Zero bound)**.** *Let $t \in \mathbb{R}$. If $\rho_1 = \sigma_1 + it_1 \in \mathcal{Z}_t$ then $\sigma_1 \leq 1$.*

Proof. By definition 486 $\rho_1 \in \mathcal{Z}_t$ implies $\zeta(\rho_1) = 0$. Now apply Lemma 490. □

Lemma 492 (Outside zeros)**.** *For $\delta > 0$ and $t \in \mathbb{R}$, let $s = 1 + \delta + it$. Then $s \notin \mathcal{Z}_t$.*

Proof. We have $\Re(s) = 1 + \delta > 1$ since $\delta > 0$. Thus $\zeta(s) \neq 0$ by theorem 468, and so $s \notin \mathcal{Z}_t$. □

Lemma 493 (Half disk)**.** *For $0 < \delta < 1$ and $t \in \mathbb{R}$, let $c = 3/2 + it$. Then $1 + \delta + it \in \mathbb{D}_{1/2}(c)$.*

Proof. We calculate $1+\delta+it-c = 1+\delta-3/2 = 1/2-\delta$. Hence $|(1+\delta+it)-c| \leq |1/2-\delta| \leq 1/2$ so $1 + \delta + it \in \mathbb{D}_{1/2}(c)$. □

Lemma 494 (Sum bound)**.** *There exists a constant $C > 0$ such that for all $0 < \delta < 1$ and $t \in \mathbb{R}$, we have*

$$\left| \sum_{\rho_1 \in \mathcal{Z}_t} \frac{m_{\rho_1,\zeta}}{1+\delta+it-\rho_1} - Z(1+\delta+it) \right| \leq C \log(|t|+2).$$

Proof. Apply Lemma 483 with $z = 1+\delta+it$ and $r_1 = 1/2$ and $r = 2/3$. For $c = 3/2 + it$, note $z \in \mathbb{D}_{r_1}(c)$ by theorem 493. Further $\mathcal{Z}_t = \mathcal{K}_\zeta(5/6; c)$ and $z \notin \mathcal{K}_\zeta(5/6; c)$ by theorem 492. We choose $C_1 = C(\frac{1}{(r-r_1)^3} + 1)$. □

Lemma 495 (Real bound)**.** *There exists a constant $C > 0$ such that for all $0 < \delta < 1$ and $t \in \mathbb{R}$, we have*

$$\Re\left(\sum_{\rho_1 \in \mathcal{Z}_t} \frac{m_{\rho_1,\zeta}}{1+\delta+it-\rho_1} - Z(1+\delta+it) \right) \leq C \log(|t|+2).$$

Proof. Apply Lemma 494 and use Mathlib: Complex.re_le_abs $\Re(w) \leq |w|$ for $w = \sum_{\rho_1 \in \mathcal{Z}_t} \frac{m_{\rho_1,\zeta}}{1+\delta+it-\rho_1} - Z(1+\delta+it)$. □

Lemma 496 (Split real)**.** *There exists a constant $C > 0$ such that for all $0 < \delta < 1$ and $t \in \mathbb{R}$, we have*

$$\Re\left(\sum_{\rho_1 \in \mathcal{Z}_t} \frac{m_{\rho_1,\zeta}}{1+\delta+it-\rho_1} \right) + \Re\left(-Z(1+\delta+it)\right) \leq C \log(|t|+2).$$

Proof. □

Lemma 497 (Double real)**.** *There exists a constant $C > 0$ such that for all $0 < \delta < 1$ and $t \in \mathbb{R}$, we have*

$$\Re\left(\sum_{\rho_1 \in \mathcal{Z}_{2t}} \frac{m_{\rho_1,\zeta}}{1+\delta+2it-\rho_1} \right) + \Re\left(-Z(1+\delta+2it)\right) \leq C \log(|2t|+2).$$

Proof. Apply Lemma 495 with $2t$. □

Lemma 498 (Real sum)**.** *If $\mathcal{Z}$ is a finite set and $c_z \in \mathbb{C}$ for $z \in \mathcal{Z}$, then $\Re(\sum_{z \in \mathcal{Z}} c_z) = \sum_{z \in \mathcal{Z}} \Re(c_z)$.*

Proof. □

Lemma 499 (Sum split)**.** *For $t \in \mathbb{R}$ and $0 < \delta < 1$, we have*

$$\Re\Big(\sum_{\rho_1 \in \mathcal{Z}_t} \frac{m_{\rho_1,\zeta}}{1+\delta+it-\rho_1} \Big) = \sum_{\rho_1 \in \mathcal{Z}_t} \Re\Big(\frac{m_{\rho_1,\zeta}}{1+\delta+it-\rho_1} \Big)$$

Proof. Apply Lemmas 487 and 498 with $\mathcal{Z} = \mathcal{Z}_t$, $z = \rho_1$, and $c_z = \frac{m_{\rho_1,\zeta}}{1+\delta+it-z}$. □

Lemma 500 (Sum split)**.** *For $t \in \mathbb{R}$ and $0 < \delta < 1$, we have*

$$\Re\Big(\sum_{\rho_1 \in \mathcal{Z}_{2t}} \frac{m_{\rho_1,\zeta}}{1+\delta+2it-\rho_1} \Big) = \sum_{\rho_1 \in \mathcal{Z}_{2t}} \Re\Big(\frac{m_{\rho_1,\zeta}}{1+\delta+2it-\rho_1} \Big)$$

Proof. Apply Lemma 499 with $2t$. □

Lemma 501 (Difference form). *For $\delta > 0$, $t \in \mathbb{R}$, and $\rho_1 = \sigma_1 + it_1 \in \mathcal{Z}_t$ we have*

$$1 + \delta + it - \rho_1 = (1 + \delta - \sigma_1) + i(t - t_1).$$

Proof. We calculate $1 + \delta + it - \rho_1 = 1 + \delta + it - (\sigma_1 + it_1) = (1 + \delta - \sigma_1) + i(t - t_1)$. □

Lemma 502 (Real part). *For $\delta > 0$, $t \in \mathbb{R}$, and $\rho_1 = \sigma_1 + it_1 \in \mathcal{Z}_t$ we have*

$$\Re(1 + \delta + it - \rho_1) = 1 + \delta - \sigma_1.$$

Proof. Apply Lemma 501, then take the real part. □

Lemma 503 (Delta bound). *For $\delta > 0$, $t \in \mathbb{R}$, and $\rho_1 = \sigma_1 + it_1 \in \mathcal{Z}_t$ we have*

$$1 + \delta - \sigma_1 \geq \delta.$$

Proof. Apply Lemma 491. □

Lemma 504 (Real delta). *For $\delta > 0$, $t \in \mathbb{R}$, and $\rho_1 = \sigma_1 + it_1 \in \mathcal{Z}_t$ we have*

$$\Re(1 + \delta + it - \rho_1) \geq \delta.$$

Proof. Apply Lemmas 502 and 503. □

Lemma 505 (Positive real). *For $\delta > 0$, $t \in \mathbb{R}$, and $\rho_1 = \sigma_1 + it_1 \in \mathcal{Z}_t$ we have*

$$\Re(1 + \delta + it - \rho_1) > 0.$$

Proof. Apply Lemma 504 and $\delta > 0$. □

Lemma 506 (Inverse real). *For $\delta > 0$, $t \in \mathbb{R}$, and $\rho_1 = \sigma_1 + it_1 \in \mathcal{Z}_t$ we have*

$$\Re\Big(\frac{1}{1 + \delta + it - \rho_1}\Big) \geq 0$$

Proof. Apply Lemmas 505 and 488 with $z = 1 + \delta + it - \rho_1$. □

Lemma 507 (Scaled real). *For $0 < \delta < 1$, $t \in \mathbb{R}$, and $\rho_1 = \sigma_1 + it_1 \in \mathcal{Z}_t$ we have*

$$\Re\Big(\frac{m_{\rho_1,\zeta}}{1 + \delta + it - \rho_1}\Big) \geq 0$$

Proof. Apply theorem 506 and Complex.re_nsmul with $n = m_{\rho_1,\zeta}$. Note $m_{\rho_1,\zeta} \in \mathbb{N}$ by 251. □

Lemma 508 (Double real). *For $0 < \delta < 1$, $t \in \mathbb{R}$, and $\rho_1 = \sigma_1 + it_1 \in \mathcal{Z}_{2t}$ we have*

$$\Re\Big(\frac{m_{\rho_1,\zeta}}{1 + \delta + 2it - \rho_1}\Big) \geq 0$$

Proof. Apply Lemma 507 with $2t$. □

Lemma 509 (Sum nonneg). *For $t \in \mathbb{R}$ and $0 < \delta < 1$, we have*

$$\sum_{\rho_1 \in \mathcal{Z}_{2t}} \Re\Big(\frac{m_{\rho_1,\zeta}}{1 + \delta + 2it - \rho_1}\Big) \geq 0$$

Proof. Apply Lemma 508. ☐

Lemma 510 (Real nonneg). *For $t \in \mathbb{R}$ and $0 < \delta < 1$, we have*

$$\Re\Big(\sum_{\rho_1 \in \mathcal{Z}_{2t}} \frac{m_{\rho_1,\zeta}}{1+\delta+2it-\rho_1}\Big) \geq 0$$

Proof. Apply Lemmas 500 and 509. ☐

Lemma 511 (Double bound). *There exists a constant $C > 0$ such that for all $0 < \delta < 1$ and $t \in \mathbb{R}$, we have*

$$\Re\left(-Z(1+\delta+2it)\right) \leq C \log(|2t|+2).$$

Proof. Apply Lemmas 497 and 510. ☐

Lemma 512 (Log compare). *For $t \geq 2$ we have $O(\log(2t)) \leq O(\log t)$*

Proof. Apply Lemmas 1 and 4. ☐

Lemma 513 (Trivial bound). *For $t \in \mathbb{R}$ we have $|2t| + 2 \geq 0$.*

Proof. ☐

Lemma 514 (Log compare). *For $t \in \mathbb{R}$ we have $O(\log(|2t|+4)) \leq O(\log(|t|+2))$*

Proof. Apply Lemmas 513 and 512 with $w = |t| + 2$. ☐

Lemma 515 (Shift bound). *There exists a constant $C > 0$ such that for all $0 < \delta < 1$ and $t \in \mathbb{R}$, we have*

$$\Re\left(-Z(1+\delta+2it)\right) \leq C \log(|t|+2).$$

Proof. Apply Lemmas 511 and 514. ☐

Lemma 516 (Split sum). *For $0 < \delta < 1$, $\sigma, t \in \mathbb{R}$, and $\rho = \sigma + it \in \mathcal{Z}$ we have*

$$\sum_{\rho_1 \in \mathcal{Z}_t} \Re\Big(\frac{m_{\rho_1,\zeta}}{1+\delta+it-\rho_1}\Big) = \Re\Big(\frac{m_{\rho,\zeta}}{1+\delta+it-\rho}\Big) + \sum_{\rho_1 \in \mathcal{Z}_t, \rho_1 \neq \rho} \Re\Big(\frac{m_{\rho_1,\zeta}}{1+\delta+it-\rho_1}\Big).$$

Proof. Apply Lemma 527. ☐

Lemma 517 (Split bound). *For $0 < \delta < 1$, $\sigma, t \in \mathbb{R}$, and $\rho = \sigma + it \in \mathcal{Z}$ we have*

$$\sum_{\rho_1 \in \mathcal{Z}_t} \Re\Big(\frac{m_{\rho_1,\zeta}}{1+\delta+it-\rho_1}\Big) \geq \Re\Big(\frac{1}{1+\delta+it-\rho}\Big).$$

Proof. Apply Lemmas 516 and 506. Note $m_{\rho_1,\zeta} \geq 1$ by theorem 252. ☐

Lemma 518 (Difference real). *For $\delta > 0$, $\sigma, t \in \mathbb{R}$, and $\rho = \sigma + it \in \mathcal{Z}$ we have*

$$1 + \delta + it - \rho = 1 + \delta - \sigma.$$

Proof. We calculate $1 + \delta + it - \rho = 1 + \delta + it - (\sigma + it) = 1 + \delta - \sigma$. ☐

Lemma 519 (Real inverse). *For $0 < \delta < 1$, $\sigma, t \in \mathbb{R}$, and $\rho = \sigma + it \in \mathcal{Z}$ we have*

$$\Re\Big(\frac{1}{1+\delta+it-\rho}\Big) = \Re\Big(\frac{1}{1+\delta-\sigma}\Big)$$

Proof. Apply Lemma 518. ☐

Lemma 520 (Inverse real)**.** *For $0 < \delta < 1$, $\sigma \leq 1$ we have $\frac{1}{1+\delta-\sigma} \in \mathbb{R}$.*

Proof. We calculate $1 + \delta - \sigma \geq \delta > 0$. Thus $\frac{1}{1+\delta-\sigma} \in \mathbb{R}$. ☐

Lemma 521 (Inverse real)**.** *For $\delta > 0$, $\sigma, t \in \mathbb{R}$, and $\rho = \sigma + it \in \mathcal{Z}$ we have $\frac{1}{1+\delta-\sigma} \in \mathbb{R}$.*

Proof. Apply Lemmas 490 and 520. ☐

Lemma 522 (Real part)**.** *For $x \in \mathbb{R}$ we have $\Re(x) = x$.*

Proof. ☐

Lemma 523 (Real inverse)**.** *For $\delta > 0$, $\sigma, t \in \mathbb{R}$, and $\rho = \sigma + it \in \mathcal{Z}$ we have*

$$\Re\Big(\frac{1}{1 + \delta - \sigma}\Big) = \frac{1}{1 + \delta - \sigma}$$

Proof. Apply Lemmas 521 and 522 with $x = \frac{1}{1+\delta-\sigma}$. ☐

Lemma 524 (Real inverse)**.** *For $\delta > 0$, $\sigma, t \in \mathbb{R}$, and $\rho = \sigma + it \in \mathcal{Z}$ we have*

$$\Re\Big(\frac{1}{1 + \delta + it - \rho}\Big) = \frac{1}{1 + \delta - \sigma}$$

Proof. Apply Lemmas 519 and 523 ☐

Lemma 525 (Sum bound)**.** *For $0 < \delta < 1$, $\sigma, t \in \mathbb{R}$, and $\rho = \sigma + it \in \mathcal{Z}$ we have*

$$\sum_{\rho_1 \in \mathcal{Z}_t} \Re\Big(\frac{m_{\rho_1,\zeta}}{1 + \delta + it - \rho_1}\Big) \geq \frac{1}{1 + \delta - \sigma}.$$

Proof. Apply Lemmas 517 and 524. ☐

Lemma 526 (Real bound)**.** *For $0 < \delta < 1$, $\sigma, t \in \mathbb{R}$, and $\rho = \sigma + it \in \mathcal{Z}$ we have*

$$\Re\Big(\sum_{\rho_1 \in \mathcal{Z}_t} \frac{m_{\rho_1,\zeta}}{1 + \delta + it - \rho_1}\Big) \geq \frac{1}{1 + \delta - \sigma}.$$

Proof. Apply Lemmas 499 and 525. ☐

Lemma 527 (In set)**.** *For $\delta > 0$, $t \in \mathbb{R}$, let $\rho = 1 + \delta + it$. If $\rho \in \mathcal{Z}$, then $\rho \in \mathcal{Z}_t$.*

Proof. Let $c = 3/2 + it$. Then we calculate $|\rho - c| = |1 + \delta - 3/2| = |1/2 - \delta| \leq 1/2$. And since $\rho \in \mathcal{Z}$, we have $\zeta(\rho) = 0$. Thus $|\rho - c| \leq 1/2$ and $\zeta(\rho) = 0$ together imply $\rho \in \mathcal{Z}_t$. ☐

Lemma 528 (Zeta bound)**.** *There exists a constant $C > 1$ such that for all $0 < \delta < 1$ and $t \in \mathbb{R}$ and $\rho = \sigma + it \in \mathcal{Z}$, we have*

$$\Re\Big(-Z(1 + \delta + it)\Big) \leq -\frac{1}{1 + \delta - \sigma} + C\log(|t| + 2).$$

Proof. Apply Lemma 527 so that $\rho \in \mathcal{Z}$ implies $\rho \in \mathcal{Z}_t$. Then apply 496 and 526. ☐

Lemma 529 (At one). *For $\delta > 0$ we have*

$$-Z(1+\delta) = \frac{1}{\delta} + O(1).$$

Proof. □

Lemma 530 (Real one). *For $\delta > 0$ we have*

$$\Re\Big(-Z(1+\delta)\Big) = \frac{1}{\delta} + O(1).$$

Proof. Apply Lemma 529. □

Lemma 531 (One bound). *There exists a constant $C > 0$ such that for all $\delta > 0$ we have*

$$\Big|Z(1+\delta) - \frac{1}{\delta}\Big| \le C.$$

Proof. By Lemma 529. □

Lemma 532 (Real bound). *There exists a constant $C > 0$ such that for all $\delta > 0$ we have*

$$\Re\Big(-Z(1+\delta) - \frac{1}{\delta}\Big) \le C.$$

Proof. By Lemma 531 and Mathlib Complex.re_le_abs for $z = Z(1+\delta) + \frac{1}{\delta}$. □

Lemma 533 (Real sum). *There exists a constant $C > 0$ such that for all $\delta > 0$ we have*

$$\Re(-Z(1+\delta)) + \Re(-\frac{1}{\delta}) \le C.$$

Proof. By Lemma 532 and Mathlib: Complex.add_re with $z = -Z(1+\delta)$ and $w = -\frac{1}{\delta}$. □

Lemma 534 (Real diff). *There exists a constant $C > 0$ such that for all $\delta > 0$ we have*

$$\Re(-Z(1+\delta)) - \frac{1}{\delta} \le C.$$

Proof. By Lemma 533 and Mathlib: RCLike.re_to_real with $x = 1/\delta$, since $1/\delta \in \mathbb{R}$. □

Lemma 535 (Combined bound). *There exists a constant $C > 0$ such that for all $0 < \delta < 1$ and $t \in \mathbb{R}$ with $|t| > 3$, if $\sigma + it \in \mathcal{Z}$ then*

$$3\Re(-Z(1+\delta)) + 4\Re(-Z(1+\delta+it)) + \Re(-Z(1+\delta+2it))$$
$$\le \frac{3}{\delta} - \frac{4}{1+\delta-\sigma} + C\log(|t|+2)$$

Proof. Apply Lemmas 532 and 528 and 515 □

Lemma 536 (Series form). *Let $s \in \mathbb{C}$. If $\mathrm{Re}(s) > 1$ then*

$$-Z(s) = \sum_{n=1}^{\infty} \Lambda(n)\, n^{-s}$$

Proof. Apply definition 484 for $Z(s)$ and `LSeries_vonMangoldt_eq_deriv_riemannZeta_div` from Mathlib/NumberTheory/LSeries/Dirichlet.lean □

Lemma 537 (Series form)**.** *For $x, y \in \mathbb{R}$, if $x > 1$ then*

$$-Z(x+iy) = \sum_{n=1}^{\infty} \Lambda(n)\, n^{-(x+iy)}$$

Proof. Let $s = x + iy$ so that $\mathrm{Re}(s) = x > 1$. Apply Lemma 536 with $s = x + iy$. □

Lemma 538 (Converges)**.** *Let $x, y \in \mathbb{R}$. If $x > 1$ then $Z(x + iy)$ converges.*

Proof. Apply definition 484 for $Z(x + iy)$. □

Lemma 539 (Real converges)**.** *Let $x, y \in \mathbb{R}$. If $x > 1$ then $\Re(-Z(x + iy))$ converges.*

Proof. Apply Lemma 538. □

Lemma 540 (Exponent split)**.** *For any $n \geq 1$ and $x, y \in \mathbb{R}$ we have $n^{-(x+iy)} = n^{-x}n^{-iy}$.*

Proof. Use $-(x + iy) = -x - iy$, and Lemma 5 with $\alpha = -x$ and $\beta = -iy$. □

Lemma 541 (Series split)**.** *For $x, y \in \mathbb{R}$, if $x > 1$ then*

$$-Z(x+iy) = \sum_{n=1}^{\infty} \Lambda(n)\, n^{-x} n^{-iy}$$

Proof. Apply Lemmas 537 and 540. □

Lemma 542 (Series converges)**.** *Let $x, y \in \mathbb{R}$. If $x > 1$ then $\sum_{n=1}^{\infty} \Lambda(n)\, n^{-x}n^{-iy}$ converges.*

Proof. Apply Lemmas 538 and 541. □

Lemma 543 (Real terms)**.** *For $n, x \geq 1$ we have $\Lambda(n)\, n^{-x} \geq 0$*

Proof. Proven by definition of von Mangoldt $\Lambda(n) \geq 0$ and $n^{-x} \geq 0$. □

Lemma 544 (Real sum)**.** *We have $\Re\Big(\sum_{n=1}^{\infty} \Lambda(n)\, n^{-x}n^{-iy}\Big) = \sum_{n=1}^{\infty} \Re(\Lambda(n)\, n^{-x}n^{-iy})$*

Proof. Apply Lemmas 542 and 7. □

Lemma 545 (Series real)**.** *For $x > 1$ and $y \in \mathbb{R}$, we have $\Re(-Z(x+iy)) = \sum_{n=1}^{\infty} \Re(\Lambda(n)\, n^{-x}n^{-iy})$*

Proof. Apply Lemmas 541 and 544. □

Lemma 546 (Real factor)**.** *For $x > 1$ and $y \in \mathbb{R}$, we have $\Re(\Lambda(n)\, n^{-x}n^{-iy}) = \Lambda(n)\, n^{-x}\, \Re(n^{-iy})$.*

Proof. Let $b = \Lambda(n)\, n^{-x}$. By Lemma 543 $b \in \mathbb{R}$. Apply Lemma 6 with $b = \Lambda(n)\, n^{-x}$ and $z = n^{-iy}$. □

Lemma 547 (Real series)**.** *For $x > 1$ and $y \in \mathbb{R}$, we have $\Re(-Z(x+iy)) = \sum_{n=1}^{\infty} \Lambda(n)\, n^{-x}\, \Re(n^{-iy})$*

Proof. Apply Lemmas 545 and 546. □

Lemma 548 (Cos form)**.** *For $x > 1$ and $y \in \mathbb{R}$, $\Re(-Z(x + iy)) = \sum_{n=1}^{\infty} \Lambda(n)n^{-x} \cos(y \log n)$.*

Proof. Apply Lemmas 547 and 16 □

Lemma 549 (Cos series). *For $x > 1$ and $y \in \mathbb{R}$, $\sum_{n=1}^{\infty} \Lambda(n) n^{-x} \cos(y \log n)$ converges.*

Proof. Apply Lemmas 539 and 548. □

Lemma 550 (Double cos). *For $x > 1$ and $t \in \mathbb{R}$, $\sum_{n=1}^{\infty} \Lambda(n) n^{-x} \cos(2t \log n)$ converges.*

Proof. Apply Lemma 549 with $y = 2t$. □

Lemma 551 (Zero cos). *For $n \geq 1$, if $t = 0$ then $\cos(t \log n) = 1$.*

Proof. For $t = 0$, we calculate $\cos(t \log n) = \cos(0 \log n) = \cos(0) = 1$. □

Lemma 552 (Zero series). *For $x > 1$, $\sum_{n=1}^{\infty} \Lambda(n) n^{-x}$ converges.*

Proof. Apply Lemma 549 with $y = 0$, and Lemma 551. □

Lemma 553 (Delta series). *For $t \in \mathbb{R}$ and $\delta > 0$,*

$$\mathrm{Re}\Big(-Z(1+\delta+it) \Big) = \sum_{n=1}^{\infty} \Lambda(n) n^{-(1+\delta)} \cos(t \log n)$$

Proof. Apply Lemma 548 with $x = 1 + \delta$ and $y = t$. Note $x > 1$ since $\delta > 0$. □

Lemma 554 (Delta double). *For $t \in \mathbb{R}$ and $\delta > 0$, we have*

$$\mathrm{Re}\Big(-Z(1+\delta+2it) \Big) = \sum_{n=1}^{\infty} \Lambda(n) n^{-(1+\delta)} \cos(2t \log n)$$

Proof. Apply Lemma 548 with $x = 1 + \delta$ and $y = 2t$. Note $x > 1$ since $\delta > 0$. □

Lemma 555 (Delta zero). *Let $\delta > 0$. We have*

$$\mathrm{Re}\Big(-Z(1+\delta) \Big) = \sum_{n=1}^{\infty} \Lambda(n) n^{-(1+\delta)}$$

Proof. Apply Lemma 548 with $t = 0$, and Lemma 551. □

Lemma 556 (341 series). *For $t \in \mathbb{R}$ and $\delta > 0$, we have*

$$\begin{aligned} &3\Re(-Z(1+\delta)) + 4\Re(-Z(1+\delta+it)) + \Re(-Z(1+\delta+2it)) \\ &= 3\sum_{n=1}^{\infty} \Lambda(n) n^{-(1+\delta)} + 4\sum_{n=1}^{\infty} \Lambda(n) n^{-(1+\delta)} \cos(t \log n) + \sum_{n=1}^{\infty} \Lambda(n) n^{-(1+\delta)} \cos(2t \log n). \end{aligned}$$

Proof. Apply Lemmas 555 and 553 and 554. □

Lemma 557 (Sum converges). *For $t \in \mathbb{R}$ and $\delta > 0$,*

$$3\sum_{n=1}^{\infty} \Lambda(n) n^{-(1+\delta)} + 4\sum_{n=1}^{\infty} \Lambda(n) n^{-(1+\delta)} \cos(t \log n) + \sum_{n=1}^{\infty} \Lambda(n) n^{-(1+\delta)} \cos(2t \log n)$$

converges.

Proof. Apply Lemmas 552 and 549 and 550. □

Lemma 558 (Factor form)**.** *For $t \in \mathbb{R}$ and $\delta > 0$, we have*

$$3\sum_{n=1}^{\infty}\Lambda(n)n^{-(1+\delta)} + 4\sum_{n=1}^{\infty}\Lambda(n)n^{-(1+\delta)}\cos(t\log n) + \sum_{n=1}^{\infty}\Lambda(n)n^{-(1+\delta)}\cos(2t\log n)$$
$$= \sum_{n=1}^{\infty}\Lambda(n)n^{-(1+\delta)}(3 + 4\cos(t\log n) + \cos(2t\log n)).$$

Proof. Apply Lemmas 552 and 549 and 550. □

Lemma 559 (Factor conv)**.** *For $t \in \mathbb{R}$ and $\delta > 0$,*

$$\sum_{n=1}^{\infty}\Lambda(n)n^{-(1+\delta)}(3 + 4\cos(t\log n) + \cos(2t\log n))$$

converges.

Proof. Apply Lemmas 557 and 558. □

Lemma 560 (Series equal)**.** *For $t \in \mathbb{R}$ and $\delta > 0$, we have*

$$3\Re(-Z(1+\delta)) + 4\Re(-Z(1+\delta+it)) + \Re(-Z(1+\delta+2it))$$
$$= \sum_{n=1}^{\infty}\Lambda(n)n^{-(1+\delta)}(3 + 4\cos(t\log n) + \cos(2t\log n)).$$

Proof. Apply Lemmas 556 and 558 □

Lemma 561 (Term nonneg)**.** *For $n \geq 1$, $\delta > 0$, and $t \in \mathbb{R}$, we have $0 \leq \Lambda(n)\,n^{-(1+\delta)}(3 + 4\cos(t\log n) + \cos(2t\log n))$.*

Proof. Apply Lemmas 25 and 543 with $x = 1 + \delta$. □

Lemma 562 (Series nonneg)**.** *For $t \in \mathbb{R}$ and $\delta > 0$, we have*

$$0 \leq \sum_{n=1}^{\infty}\Lambda(n)\,n^{-(1+\delta)}(3 + 4\cos(t\log n) + \cos(2t\log n))$$

Proof. Apply Lemmas 559, 561, 25, and 26 with $r_n = \Lambda(n)\,n^{-(1+\delta)}(3+4\cos(t\log n)+\cos(2t\log n))$. □

Lemma 563 (Positive sum)**.** *For $t \in \mathbb{R}$ and $\delta > 0$, we have*

$$0 \leq 3\Re(-Z(1+\delta)) + 4\Re(-Z(1+\delta+it)) + \Re(-Z(1+\delta+2it))$$

Proof. Apply Lemmas 560 and 562. □

Lemma 564 (Inequality)**.** *There exists a constant $C > 1$ such that, for any $\sigma + it \in \mathcal{Z}$,*

$$\frac{4}{1 - \sigma + 1/(2C\log(|t|+2))} \leq 7C\log(|t|+2)$$

Proof. Apply Lemmas 535 and 563 with $\delta = 1/(2C\log(|t|+2))$. □

Lemma 565 (Rearranged)**.** *There exists a constant* $C > 0$ *such that, for any* $\sigma + it \in \mathcal{Z}$*,*

$$1 - \sigma + 1/(2C \log(|t| + 2)) \geq 4/(7C \log(|t| + 2))$$

Proof. Apply Lemma 564. □

Lemma 566 (Final bound)**.** *There exists a constant* $C > 0$ *such that, for any* $\sigma + it \in \mathcal{Z}$*,*

$$1 - \sigma \geq 1/(14C \log(|t| + 2))$$

Proof. Apply Lemma 565. □

Lemma 567 (Zero free)**.** *There exists a constant* $1 > c > 0$ *such that if* $\zeta(\sigma + it) = 0$ *and* $|t| > 3$ *for some* $\sigma, t \in \mathbb{R}$*, then* $\sigma \leq 1 - \frac{c}{\log(|t|+2)}$*.*

Proof. Apply Lemma 566 with $c = 1/(14C)$, and Definition 485 of $\mathcal{Z}$. □

4.1 Bound on ζ'/ζ

Definition 568 (Delta zeros)**.** For $t \in \mathbb{R}$ and $0 < \delta < 1/9$, define

$$\mathcal{Y}_t(\delta) = \{\rho_1 \in \mathbb{C} : \zeta(\rho_1) = 0 \,\text{and}\, |\rho_1 - (1 - \delta + it)| \leq 2\delta\}.$$

Definition 569 (Delta def)**.** Let $0 < a < 1$ be the constant in 567. For $z \in \mathbb{C}$ with $|\Im(z)| > 2$, define the function $\delta(z) = \frac{a/20}{\log(|\Im(z)|+2)}$. For $t \in \mathbb{R}$ define $\delta_t = \delta(it)$.

Lemma 570 (Delta range)**.** *For* $z \in \mathbb{C}$ *we have* $0 < \delta(z) < 1/9$*. For* $t \in \mathbb{R}$ *we have* $0 < \delta_t < 1/9$*.*

Proof. Unfold theorem 569. □

Lemma 571 (Zero free)**.** *For* $z \in \mathbb{C}$*, if* $\Re(z) > 1 - 9\delta(z)$ *then* $\zeta(z) \neq 0$*.*

Proof. Unfold definition of $\delta(z)$ in theorem 569, and apply contrapositive of theorem 567. □

Lemma 572 (Disk inclusion)**.** *Let* $t \in \mathbb{R}$ *with* $|t| > 3$*. For* $c = 3/2 + it$ *and* $z = \sigma + it$ *with* $1 - \delta_t \leq \sigma \leq 3/2$*, we have* $z \in \overline{\mathbb{D}}_{2/3}(c)$*.*

Proof. We calculate $|z - c| = |\sigma - 3/2| \leq 1/2 + \delta_t$. Note $\delta_t \leq 1/9$ by theorem 570. Hence $|z - c| \leq 2/3$. □

Lemma 573 (Not zero)**.** *Let* $t \in \mathbb{R}$ *with* $|t| > 3$*. For* $c = 3/2 + it$ *and* $z = \sigma + it$ *with* $1 - \delta_t \leq \sigma \leq 3/2$*, we have* $z \notin \mathcal{K}_\zeta(5/6; c)$*.*

Proof. Since $\Re(z) = \sigma \geq 1 - \delta_t = 1 - \delta(z)$, we have $\zeta(z) \neq 0$ by theorem 571. Thus $z \notin \mathcal{K}_\zeta(5/6; c)$. □

Lemma 574 (Expansion)**.** *There exists a constant* $C_1 > 1$ *such that for all* $t \in \mathbb{R}$ *with* $|t| > 3$*, letting* $c = 3/2 + it$*, and all* $z = \sigma + it$ *with* $1 - \delta_t \leq \sigma \leq 3/2$ *we have*

$$\left| \frac{\zeta'(z)}{\zeta(z)} - \sum_{\rho \in \mathcal{K}_\zeta(5/6;c)} \frac{m_{\rho,\zeta}}{z - \rho} \right| \leq C_1 \log |t|$$

Proof. Apply theorem 483 with $z = \sigma + it$, $r_1 = 2/3$, $r = 3/4$, and choosing $C_1 = C\left(\frac{1}{(r-r_1)^3} + 1\right)$. Note $z \in \overline{\mathbb{D}}_{r_1}(c) \setminus \mathcal{K}_\zeta(5/6; c)$ by theorems 572 and 573. □

Lemma 575 (Distance real). *For $z, \rho \in \mathbb{C}$ we have $|z - \rho| \geq \Re(z) - \Re(\rho)$*

Proof. Apply Mathlib: Complex.re_le_abs and then Complex.sub_re to calculate $|z - \rho| \geq \Re(z - \rho) = \Re(z) - \Re(\rho)$. □

Lemma 576 (Real bound). *For $t \in \mathbb{R}$ with $|t| > 3$ and $\rho \in \mathcal{K}_\zeta(5/6; 3/2 + it)$ we have $\Re(\rho) \leq 1 - 9\delta(\rho)$.*

Proof. By definition $\rho \in \mathcal{K}_\zeta(5/6; 3/2 + it)$ implies $\zeta(\rho) = 0$. Then $\zeta(\rho) = 0$ implies $\Re(\rho) \leq 1 - 9\delta(\rho)$ by the contrapositive of theorem 571. □

Lemma 577 (Imag bound). *For $t \in \mathbb{R}$ with $|t| > 3$ and $z \in \overline{\mathbb{D}}_{5/6}(3/2 + it)$, we have $|\Im(z)| \leq |t| + 5/6$.*

Proof. Unfold definition of $\overline{\mathbb{D}}_{2\delta_t}(1 - \delta_t + it)$. □

Lemma 578 (Imag growth). *For $t \in \mathbb{R}$ with $|t| > 3$ and $z \in \overline{\mathbb{D}}_{5/6}(3/2 + it)$, we have $|\Im(z)| + 2 \leq (|t| + 2)^3$.*

Proof. Apply theorem 577 and $3 < |t|$. □

Lemma 579 (Log bound). *For $t \in \mathbb{R}$ with $|t| > 3$ and $z \in \overline{\mathbb{D}}_{5/6}(3/2 + it)$, we have $\log(|\Im(z)| + 2) \leq 3\log(|t| + 2)$.*

Proof. Apply theorem 578 and Mathlib: Real.log_le_log □

Lemma 580 (Log compare). *For $t \in \mathbb{R}$ with $|t| > 3$ and $z \in \overline{\mathbb{D}}_{5/6}(3/2 + it)$, we have $1/\log(|t| + 2) \leq 3/\log(|\Im(z)| + 2)$.*

Proof. Apply theorem 579 and Mathlib: one_div_le_one_div □

Lemma 581 (Delta compare). *For $t \in \mathbb{R}$ with $|t| > 3$ and $z \in \overline{\mathbb{D}}_{5/6}(3/2 + it)$, we have $\delta_t \leq 3\delta(z)$.*

Proof. Unfold definitions of δ_t and $\delta(z)$ from theorem 569. Then apply theorem 580. □

Lemma 582 (Delta bound). *Let $t \in \mathbb{R}$ with $|t| > 3$, and $c = 3/2 + it$. For all $\rho \in \mathcal{K}_\zeta(5/6; c)$ we have $\delta(\rho) \geq \frac{1}{3}\delta_t$*

Proof. Apply theorem 581 with $z = \rho$. Note $\mathcal{K}_\zeta(5/6; c) \subset \overline{\mathbb{D}}_{5/6}(c)$. □

Lemma 583 (Real bound). *Let $t \in \mathbb{R}$ with $|t| > 3$, and $c = 3/2 + it$. For all $\rho \in \mathcal{K}_\zeta(5/6; c)$ we have $\Re(\rho) \leq 1 - 3\delta_t$*

Proof. Apply theorems 576 and 582. □

Lemma 584 (Gap bound). *Let $t \in \mathbb{R}$ with $|t| > 3$, and $c = 3/2 + it$. For all $\rho \in \mathcal{K}_\zeta(5/6; c)$ and $z = \sigma + it$ with $1 - \delta_t \leq \sigma \leq 3/2$ we have $\Re(z) - \Re(\rho) \geq 2\delta_t$.*

Proof. Apply theorem 583, and calculate $\Re(z) - \Re(\rho) \geq (1 - \delta_t) - (1 - 3\delta_t) = 2\delta_t$. □

Lemma 585 (Gap size). *Let $t \in \mathbb{R}$ with $|t| > 3$, and $c = 3/2 + it$. For all $\rho \in \mathcal{K}_\zeta(5/6; c)$ and $z = \sigma + it$ with $1 - \delta_t \leq \sigma \leq 3/2$ we have $|z - \rho| \geq 2\delta_t$.*

Proof. Apply theorems 575 and 584 □

Lemma 586 (Nonzero gap)**.** *Let $t \in \mathbb{R}$ with $|t| > 3$, and $c = 3/2 + it$. For all $\rho \in \mathcal{K}_\zeta(5/6; c)$ and $z = \sigma + it$ with $1 - \delta_t \leq \sigma \leq 3/2$ we have $|z - \rho| > 0$.*

Proof. Apply theorems 570 and 585. □

Lemma 587 (Inverse gap)**.** *Let $t \in \mathbb{R}$ with $|t| > 3$, and $c = 3/2 + it$. For all $\rho \in \mathcal{K}_\zeta(5/6; c)$ and $z = \sigma + it$ with $1 - \delta_t \leq \sigma \leq 3/2$ we have $\frac{1}{|z-\rho|} \leq \frac{1}{2\delta_t}$.*

Proof. Apply theorem 585 and Mathlib: one_div_le_one_div with theorem 570. □

Lemma 588 (Order nat)**.** *For $t \in \mathbb{R}$ with $|t| > 3$, let $c = 3/2 + it$. Then $m_{\rho,\zeta} \in \mathbb{N}$ for all $\rho \in K_\zeta(5/6; c)$.*

Proof. Apply theorems 251 and 474 with ζ, $R_1 = 5/6$, $R = 8/9$. Note ζ AnalyticOnNhd $\overline{\mathbb{D}}_1(c)$ by theorem 467. Also $\zeta(c) \neq 0$ by theorem 469 □

Lemma 589 (Finite set)**.** *For $t \in \mathbb{R}$ with $|t| > 3$, let $c = 3/2 + it$. Then $K_\zeta(5/6; c)$ is finite.*

Proof. Apply theorems 249 and 473 with ζ, $R_1 = 5/6$, $R = 8/9$. Note ζ AnalyticOnNhd $\overline{\mathbb{D}}_1(c)$ by theorem 467. Also $\zeta(c) \neq 0$ by theorem 469 □

Lemma 590 (Triangle)**.** *For all $t \in \mathbb{R}$ with $|t| > 3$, letting $c = 3/2 + it$, and all $z = \sigma + it$ with $1 - \delta_t \leq \sigma \leq 3/2$ we have*

$$\left| \sum_{\rho \in \mathcal{K}_\zeta(5/6;c)} \frac{m_{\rho,\zeta}}{z - \rho} \right| \leq \sum_{\rho \in \mathcal{K}_\zeta(5/6;c)} \frac{m_{\rho,\zeta}}{|z - \rho|}$$

Proof. Apply Mathlib: Finset.abs_sum_le_sum_ab. Note $\mathcal{K}_\zeta(5/6; c)$ is finite by theorem 589. Then apply Mathlib: abs_div with theorem 586. Note $m_{\rho,\zeta} \in \mathbb{N}$ by theorem 588. □

Lemma 591 (Triangle)**.** *There exists a constant $C_1 > 1$ such that for all $t \in \mathbb{R}$ with $|t| > 3$, letting $c = 3/2 + it$, and all $z = \sigma + it$ with $1 - \delta_t \leq \sigma \leq 3/2$ we have*

$$\left| \frac{\zeta'(z)}{\zeta(z)} \right| \leq \sum_{\rho \in \mathcal{K}_\zeta(5/6;c)} \frac{m_{\rho,\zeta}}{|z - \rho|} + C_1 \log |t|$$

Proof. Apply theorems 574 and 590. □

Lemma 592 (Sum bound)**.** *For all $t \in \mathbb{R}$ with $|t| > 3$, letting $c = 3/2 + it$, and all $z = \sigma + it$ with $1 - \delta_t \leq \sigma \leq 3/2$ we have*

$$\sum_{\rho \in \mathcal{K}_\zeta(5/6;c)} \frac{m_{\rho,\zeta}}{|z - \rho|} \leq \frac{1}{2\delta_t} \sum_{\rho \in \mathcal{K}_\zeta(5/6;c)} m_{\rho,\zeta}$$

Proof. Apply theorem 587. □

Lemma 593 (Order bound)**.** *Let $B > 1$, $0 < R_1 < R < 1$, and $f : \mathbb{C} \to \mathbb{C}$ be a function that is AnalyticOnNhd $\overline{\mathbb{D}}_1(c)$. If $f(c) \neq 0$ and $|f(z)| \leq B$ on $z \in \overline{\mathbb{D}}_R(c)$, then $\sum_{\rho \in \mathcal{K}_f(R_1;c)} m_{\rho,f} \leq \frac{\log(B/|f(c)|)}{\log(R/R_1)}$.*

Proof. Use the conditions from theorems 472 to 474 with $f = \zeta$, and then apply theorem 324 with $\zeta_c(z) = \zeta(z+c)/\zeta(c)$. Note ζ is AnalyticOnNhd $\overline{\mathbb{D}}_1(c)$ by theorem 467. Also $\zeta(c) \neq 0$ by theorem 469. ☐

Lemma 594 (Order sum). *There exists a constant $C_2 > 1$ such that for all $t \in \mathbb{R}$ with $|t| > 3$, letting $c = 3/2 + it$, we have $\sum_{\rho \in \mathcal{K}_\zeta(5/6;c)} m_{\rho,\zeta} \leq C_2 \log|t|$*

Proof. Apply theorem 593 with $R_1 = 5/6$, $R = 8/9$. Then by theorem 481, we set $B = b|t|$. Thus $\log(B/|f(c)|) \leq \log|t| + \log(b/|f(c)|)$. Thus we may choose $C_2 = 2(1 + \log(b/|f(c)|))/\log(R/R_1)$. ☐

Lemma 595 (Sum bound). *There exists a constant $C_3 > 1$ such that for all $t \in \mathbb{R}$ with $|t| > 3$, letting $c = 3/2 + it$, and all $z = \sigma + it$ with $1 - \delta_t \leq \sigma \leq 3/2$ we have*

$$\sum_{\rho \in \mathcal{K}_\zeta(5/6;c)} \frac{m_{\rho,\zeta}}{|z - \rho|} \leq \frac{C_3}{\delta_t} \log|t|$$

Proof. Apply theorems 592 and 594. ☐

Lemma 596 (Sum bound). *There exists a constant $C_4 > 1$ such that for all $t \in \mathbb{R}$ with $|t| > 3$, letting $c = 3/2 + it$, and all $z = \sigma + it$ with $1 - \delta_t \leq \sigma \leq 3/2$ we have*

$$\sum_{\rho \in \mathcal{K}_\zeta(5/6;c)} \frac{m_{\rho,\zeta}}{|z - \rho|} \leq C_4 \log|t|^2$$

Proof. Apply theorems 569 and 595. ☐

Lemma 597 (Log bound). *There exists a constant $C > 1$ such that for all $t \in \mathbb{R}$ with $|t| > 3$, and all $s = \sigma + it$ with $1 - \delta_t \leq \sigma \leq 3/2$, we have*

$$\left|\frac{\zeta'}{\zeta}(s)\right| \leq C \log|t|^2$$

Proof. Apply theorems 323, 591 and 596. ☐

Lemma 598 (Log bound). *There exist constants $0 < A < 1$ and $C > 1$ such that for all $t \in \mathbb{R}$ with $|t| > 3$, and all $s = \sigma + it$ with $1 - A/\log(|t| + 2) \leq \sigma \leq 3/2$, we have*

$$\left|\frac{\zeta'}{\zeta}(s)\right| \leq C \log|t|^2$$

Proof. Apply theorems 569 and 597. ☐

Lemma 599 (Real bound). *Let $t \in \mathbb{R}$. If $z \in \overline{\mathbb{D}}_{2\delta_t}(1 - \delta_t + it)$ then $\Re(z) > 1 - 4\delta_t$*

Proof. Apply theorem 612 with $\delta = \delta_t$. ☐

Lemma 600 (Real bound). *For $t \in \mathbb{R}$ with $|t| > 3$ and $z \in \overline{\mathbb{D}}_{2\delta_t}(1 - \delta_t + it)$, we have $\Re(z) \geq 1 - 6\delta(z)$*

Proof. Apply theorems 581 and 599. ☐

Lemma 601 (In disk). *For $t \in \mathbb{R}$ with $|t| > 3$ we have $\mathcal{Y}_t(\delta_t) \subset \overline{\mathbb{D}}_{2\delta_t}(1 - \delta_t + it)$.*

Proof. Unfold definition of $\mathcal{Y}_t(\delta_t)$ in theorem 568 $\square$

Lemma 602 (Zero set)**.** *Let $t \in \mathbb{R}$ and $\delta > 0$. If $\rho_1 \in \mathcal{Y}_t(\delta)$ then $\zeta(\rho_1) = 0$.*

Proof. Unfold definition 568 for $\mathcal{Y}_t(\delta)$. $\square$

Lemma 603 (Real abs)**.** *For $w \in \mathbb{C}$ we have $|\Re(w)| \leq |w|$*

Proof. Mathlib (try Complex.abs_re_le_abs) $\square$

Lemma 604 (Real diff)**.** *Let $t \in \mathbb{R}$, $1/9 > \delta > 0$ and $z \in \mathbb{C}$. Then $|\Re(z - (1 - \delta + it))| \leq |z - (1 - \delta + it)|$*

Proof. Apply Lemma 603 with $w = z - (1 - \delta + it)$. $\square$

Lemma 605 (Real diff)**.** *Let $t \in \mathbb{R}$, $1/9 > \delta > 0$ and $z \in \mathbb{C}$. If $|z - (1 - \delta + it)| \leq \delta/2$ then $|\Re(z - (1 - \delta + it))| \leq \delta/2$.*

Proof. Apply Lemma 604. $\square$

Lemma 606 (Real diff)**.** *Let $t \in \mathbb{R}$ and $1/9 > \delta > 0$ and $z \in \mathbb{C}$. We have $\Re(z - (1 - \delta + it)) = \Re(z) - (1 - \delta)$*

Proof. $\square$

Lemma 607 (Real diff)**.** *Let $t \in \mathbb{R}$, $1/9 > \delta > 0$ and $z \in \mathbb{C}$. If $|z - (1 - \delta + it)| \leq \delta/2$ then $|\Re(z) - (1 - \delta)| \leq \delta/2$*

Proof. Apply Lemmas 605 and 606. $\square$

Lemma 608 (Neg bound)**.** *Let $a \in \mathbb{R}$ and $b > 0$. If $|a| \leq b$ then $a \geq -b$.*

Proof. Mathlib (try neg_le_of_abs_le) $\square$

Lemma 609 (Real bound)**.** *Let $1/9 > \delta > 0$ and $z \in \mathbb{C}$. If $|\Re(z) - (1 - \delta)| \leq \delta/2$ then $\Re(z) - (1 - \delta) \geq -\delta/2$*

Proof. 608 with $a = \Re(z) - (1 - \delta)$ and $b = \delta/2$. $\square$

Lemma 610 (Real bound)**.** *Let $0 < \delta < 1/9$ and $z \in \mathbb{C}$. If $|\Re(z) - (1 - \delta)| \leq 2\delta$ then $\Re(z) \geq 1 - 3\delta$*

Proof. Apply Lemma 609 and then add $1 - \delta$ to both sides. $\square$

Lemma 611 (Real bound)**.** *Let $0 < \delta < 1/9$ and $z \in \mathbb{C}$. If $|\Re(z) - (1 - \delta)| \leq 2\delta$ then $\Re(z) > 1 - 4\delta$*

Proof. Apply Lemma 610, and then use $1 - \frac{3}{2}\delta > 1 - 2\delta$, since $\delta > 0$. $\square$

Lemma 612 (Real bound)**.** *Let $t \in \mathbb{R}$, $0 < \delta < 1/9$ and $z \in \mathbb{C}$. If $|z - (1 - \delta + it)| \leq 2\delta$ then $\Re(z) > 1 - 4\delta$.*

Proof. Apply Lemmas 607 and 611. $\square$

Lemma 613 (Empty set)**.** *For $t \in \mathbb{R}$ with $|t| > 3$ we have $\mathcal{Y}_t(\delta_t) = \emptyset$.*

Proof. $\square$

Lemma 614 (Empty sum). *For any $g : \mathbb{C} \to \mathbb{C}$, if $S = \emptyset$ then*

$$\sum_{s \in S} g(s) = 0$$

Proof. Mathlib (try Mathlib.Meta.NormNum.Finset.sum_empty) □

Lemma 615 (Zero sum). *For $t \in \mathbb{R}$ with $|t| > 3$ we have*

$$\sum_{\rho_1 \in \mathcal{Y}_t(\delta_t)} \frac{m_{\rho_1,\zeta}}{1 - \delta_t + it - \rho_1} = 0.$$

Proof. Apply Lemmas 613 and 614 with $g(s) = \frac{m_{\rho_1,\zeta}}{1-\delta_t+it-s}$ and $S = \mathcal{Y}_t(\delta_t)$. □

Lemma 616 (Center bound). *For $\sigma \geq 3/2$ and $t \in \mathbb{R}$ we have $|\frac{\zeta'}{\zeta}(\sigma + it)| \leq |\frac{\zeta'}{\zeta}(\sigma)|$*

Proof. By Mathlib: ArithmeticFunction.LSeries_vonMangoldt_eq_deriv_riemannZeta_div we have $-\frac{\zeta'}{\zeta}(\sigma + it) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$.

Note this series is summable by Mathlib: ArithmeticFunction.LSeriesSummable_vonMangoldt

Then apply Mathlib: norm_tsum_le_tsum_norm so that $|\frac{\zeta'}{\zeta}(\sigma + it)| \leq \sum_{n=1}^{\infty} \frac{\Lambda(n)}{|n^s|}$.

Observe $|n^s| = n^{\Re(s)} e^{-\operatorname{Arg}(s)\Im(n)}$ by Mathlib: Complex.abs_cpow_le. Note $n \in \mathbb{R}$ so $\Im(n) = 0$ by imaginaryPart_ofReal. Thus $e^{-\operatorname{Arg}(s)\Im(n)} = e^0 = 1$. And $\Re(s) = \sigma$. Hence $|n^s| = n^\sigma$.

Thus $|\frac{\zeta'}{\zeta}(\sigma + it)| \leq \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma}$.

Again by Mathlib: ArithmeticFunction.LSeries_vonMangoldt_eq_deriv_riemannZeta_div we have$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} = -\frac{\zeta'}{\zeta}(\sigma)$.

Take absolute values to get $|\frac{\zeta'}{\zeta}(\sigma + it)| \leq |\frac{\zeta'}{\zeta}(\sigma)|$. □

Lemma 617 (Log bound). *There exists a constant $C > 1$ such that for all $t \in \mathbb{R}$ with $|t| > 3$, and all $s = \sigma + it$ with $\sigma \geq 3/2$, we have*

$$\left|\frac{\zeta'}{\zeta}(s)\right| \leq C$$

Proof. Apply theorem 616 and then theorem 531. □

Theorem 618 (Bound on ζ'/ζ). *There exist constants $0 < A < 1$ and $C > 1$ such that for any $t \in \mathbb{R}$ with $|t| > 3$ and $\sigma \geq 1 - A/\log(|t| + 2)$, we have*

$$\left|\frac{\zeta'}{\zeta}(\sigma + it)\right| \leq C \log |t|^2.$$

Proof. Apply theorems 323, 598 and 617 □

Lemma 619 (Zero-free region near 1). *There exists a constant $A \in (0, \frac{1}{2})$ such that for every real t with $|t| > 3$ and every real σ with*

$$\sigma \in [1 - A/\log |t|,\ 1),$$

we have

$$\zeta(\sigma + it) \neq 0.$$

In other words, a uniform zero-free region of the form $\Re s \geq 1 - A/\log |\Im s|$ holds for large $|\Im s|$.

Proof. □

Lemma 620 (Uniform bound on the logarithmic derivative of ζ)**.** *There exist constants $A \in (0, \frac{1}{2})$ and $C > 0$ such that for every real t with $|t| > 3$ and every real σ with*

$$\sigma \geq 1 - A/\log|t|,$$

the logarithmic derivative of the Riemann zeta function satisfies the uniform bound

$$\left|\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)}\right| \leq C \log|t|^2.$$

The constants A, C are absolute (independent of σ and t) and give a uniform control of ζ'/ζ in the stated region.

Proof. □

Chapter 5

Strong PNT

Theorem 621 (Strong PNT)**.** *We have*

$$\sum_{n \le x} \Lambda(n) = x + O(x \exp(-c(\log x)^{1/2})).$$

Proof. □